EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2003 / 2004

(JUNE/JULY'2005) (Re-repeat)

MT 202 - METRIC SPACE & RIEMANN INTEGRAL

Answer four questions only Time: Two hours

1. Define the term *metric space.*

(a) Let (X, d) be a metric space. Show that the function $d_1 : X \times X \to \Re$ defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} \qquad \forall x, y \in X$$

is a metric on X

(b) Show that the conditions

i. $d(x, y) = 0$, if and only if $x = y$, and

ii. $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$,

are not sufficient to ensure that the function $d : X \times X \to \Re$ is a metric on a nonempty set X.

2. (a) Let A be a subset of a metric space (X, d). Define the term *interior of A.* Prove that A^o, the interior of A, is the largest open set contained in A.

(b) Let A, B be any two subsets of a metric space (X, d). Prove that:

i. $A^o \cap B^o = (A \cap B)^o$,

ii. $A^o \cup B^o \subseteq (A \cup B)^o$.

Give an example to show $A^o \cup B^o \neq (A \cup B)^o$.

3. Define the term *compact set* in a metric space.

(a) Show that $[a, b]$ is a compact subset of $\Re$ with the usual metric,

(b) Let A be a compact subset of a metric space (X, d) and let $a \in X - A$. Prove that there exist open sets G and H such that $a \in G$, $A \subseteq H$ and $G \cap H = \Phi$. Hence, show that any compact subset of X is closed.

4. Let f be a function from a metric space (X, d_1) to a metric space (Y, d_2). Prove the following:

(a) If f is continuous at a point $a \in X$ and $\{x_n\}$ be a sequence in X such that $x_n \to a$ as $n \to \infty$, then $\{f(x_n)\} \to f(a)$.

(b) f is continuous if and only if the inverse image of every open set contained in Y is open in X.

(c) If f is continuous and A is a compact subset of X then $f(A)$ is compact in Y.

5. Let f be a bounded real valued function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$".

(a) With the usual notations, prove that a bounded real valued function f on $[a, b]$ is Riemann integrable if and only if for given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \epsilon.$$

(b) Prove that if f is continuous on $[a, b]$, then

i. f is Riemann integrable over $[a, b]$,

ii. the function $F : [a, b] \longrightarrow \Re$ defined by $F(x) = \int_a^x f(t)\, dt$ is differentiable on $[a, b]$ and $F'(x) = f(x) \quad \forall \ x \in [a, b]$.

6. When is an integral $\int_a^b f(x)\, dx$ said to be an improper integral of the first kind, the second kind and the third kind?

What is meant by the statement "an improper integral of the second kind is convergent"?

Discuss the convergence of the improper integral $\int_a^b \frac{dx}{(x-a)^p}$.

Test the convergence of the following:

(a) $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$,

(b) $\int_3^6 \frac{\ln x}{(x-3)^4}\, dx$.