EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2003/2004

(June/July'2005) (Repeat)

SECOND SEMESTER

MT 204 - RIEMANN INTEGRAL & SEQUENCE AND SERIES OF FUNCTIONS

Answer all questions

Time: Two hours

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1. Let f be a bounded real valued function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$".

   (a) With the usual notations, prove that a bounded real valued function f on $[a, b]$ is Riemann integrable if and only if for given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that
   $$U(P, f) - L(P, f) < \epsilon.$$

   (b) Prove that if f is continuous on $[a, b]$, then

   i. f is Riemann integrable over $[a, b]$,

   ii. the function $F : [a, b] \longrightarrow \mathbb{R}$ defined by $F(x) = \int_a^x f(t)\, dt$ is differentiable on $[a, b]$ and $F'(x) = f(x) \quad \forall \ x \in [a, b]$.

2. When is an integral $\int_a^b f(x)\, dx$ said to be an improper integral of the first kind, the second kind and the third kind?

(a) If $0 \leq f(x) \leq g(x)$ for all $[a, \infty)$ and if $f(x)$ and $g(x)$ are continuous on $[a, \infty)$.Prove that

(i) if $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ converges.

(ii) if $\int_a^\infty f(x)dx$ diverges then $\int_a^\infty g(x)dx$ diverges.

(b) Discuss the convergence of the improper integral $\int_a^b \frac{dx}{(x-a)^p}$.

(c) Test the convergence of the following:

(i) $\int_0^\infty \frac{dx}{x^2+\sqrt{x}}\, dx$;

(ii) $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x}\, dx}$.

3. Define the term "Uniform convergence" of a sequence of functions.

(a) Prove that the sequence of functions defined on E converges uniformly on E if and only if every $\epsilon > 0$ there exists an integer N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$ and for all $m, n \geq N$.

(b) Let $\{f_n\}$ be a sequence of functions that are integrable on $[a, b]$ and suppose that $\{f_n\}$ converges uniformly on $[a, b]$ to f. Prove that f is integrable and $\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx$.

(c) Provide a sequence of functions $\{g_n\}$ converges to a function g on an interval $[0, 1]$ such that $\int_0^1 g_n(x)\, dx$ and $\int_0^1 g(x)\, dx$ exist and $\lim_{n\to\infty} \int_0^1 g_n(x)\, dx \neq \int_0^1 g(x)\, dx$.

4. (a) Let $\{f_n\}$ be a sequence of real valued functions defined on $E \subseteq \mathbb{R}$. Suppose that for each $n \in \mathbb{N}$, there is a constant M_n such that

$$|f_n(x)| \leq M_n, \qquad \text{for all } \in E$$

where $\sum M_n$ converges. Prove that $\sum f_n$ converges uniformly on E.

(b) Let $\{f_n\}$, $\{g_n\}$ be two sequences of functions defined over a non empty set $E \subseteq \mathbb{R}$. Suppose also that

i. $|S_n| = |\sum_{k=1}^{n} f_k(x)| \leq M$ for all $x \in E$, all $n \in \mathbb{N}$.

ii. $\sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$ converges uniformly in E.

iii. $g_n \longrightarrow 0$ uniformly in E.

Prove that $\sum_{k=1}^{\infty} f_k(x) g_k(x)$ converges uniformly in E.

(c) Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n + ax^2}$ where $a > 0$ converges uniformly in $\mathbb{R}$.