

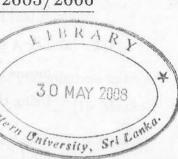
## EASTERN UNIVERSITY, SRI LANKA FIRST EXAMINATION IN SCIENCE 2005/2006

March/April' 2008

SECOND SEMESTER

MT 102 - REAL ANALYSIS

Proper & Repeat



Answer all questions

Time:Three hours

- Q1. (a) Define the terms Supremum and Infimum of a bounded subset of  $\mathbb{R}$ . [10 Marks]
  - (b) Prove that an upper bound u of a non-empty set S in  $\mathbb{R}$  is the supremum of S if and only if for each  $\varepsilon > 0$  there exists  $s_{\varepsilon} \in S$  such that  $u \varepsilon < s_{\varepsilon}$ .

    [25 Marks]
  - (c) Let A and B be subsets of  $\mathbb{R}$  that are bounded, and let  $A+B=\{a+b:a\in A,\ b\in A\}$ . Prove that

$$\sup(A+B) = \sup A + \sup B.$$

[30 Marks]

- (d) Let X be a non-empty set, and let f and g be two functions defined on X and have bounded ranges in R. Show that sup{f(x) + g(x) : x ∈ X} ≤ sup{f(x) : x ∈ X} + sup{g(x) : x ∈ X}. Give examples to show that this inequality can be either equality or strict inequality.
  [35 Marks]
- Q2. (a) Give the formal definition of the notion of a sequence of real numbers converging to a limit. Use this definition to prove that the sequence  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ ...... converges to a limit, which you should state. [30 Marks]

- (b) Use the definition of a limit to show that if  $x_n$  and  $y_n$  are sequences with  $x_n \longrightarrow x$  as  $n \longrightarrow \infty$  and  $y_n \longrightarrow y$  as  $n \longrightarrow \infty$  then  $x_n + y_n \longrightarrow x + y$  as  $n \longrightarrow \infty$ . [25 Marks
- (c) Prove that, every convergent sequence of real numbers is bounded. [30 Marks
- (d) Is it true that every bounded sequence of real numbers is convergent? Justify your answer. [15 Marks
- Q3. (a) Let  $A \subseteq \mathbb{R}$  with  $x_0 \in A$  and let  $f: A \longrightarrow \mathbb{R}$  be a function. Define what is means to say that the limit of f at  $x_0$  is  $\ell$ . (ie.,  $\lim_{x \to x_0} f(x) = \ell$ ) [10 Marks] Using the definition, Show that

$$\lim_{x \to 2} \left( \frac{x^3 - 4}{x^2 + 1} \right) = \frac{4}{5} \quad \text{and} \quad \lim_{x \to 0} \left( \frac{x^2}{|x|} \right) = 0 \quad (x \neq 0)$$
 [40 Marks]

- (b) Let  $f: \mathbb{R} \longrightarrow \mathbb{R}$  be a function and  $c \in \mathbb{R}$ . Prove that the following conditions are equivalent:
  - i.  $\lim_{x \to c} f(x) = \ell$  exists finitely.
  - ii. For every sequence  $(x_n)$  in  $\mathbb{R}$  that converges to c such that  $x_n \neq c$  for all  $n \in \mathbb{N}$ , the sequence  $(f(x_n))$  converges to  $\ell$ . [50 Marks]
- Q4. (a) Give the formal definition of what it means for a real-valued function f to be continuous at a point 'a' in its domain. [15 Marks]

  Prove that a function  $f: \mathbb{R} \longrightarrow \mathbb{R}$  defined by  $f(x) = \cos x$ ,  $\forall x \in \mathbb{R}$  is
  - continuous at every point in R. [20 Marks]
  - (b) Let I = [a, b] be a closed and bounded interval with a < b and let  $f : I \longrightarrow \mathbb{R}$  be continuous on I. Prove that f is bounded on I. [35 Marks]
  - (c) State the *Intermediate Value Theorem* and use it to show that the equation  $2x^2(x+2) 1 = 0$  has a root in each of the intervals (-2, -1), (-1, 0) and (0, 1).
- Q5. (a) Give the definition of a function  $f:[a,b] \longrightarrow \mathbb{R}$  being differentiable at  $c \in (a,b)$ . Define  $f:[0,\infty) \longrightarrow \mathbb{R}$  as  $f(x) = \sqrt{x}$ , for  $x \ge 0$ . Using the definition, show that  $f'(c) = \frac{1}{2\sqrt{c}}$ , for c > 0. [35 Marks]
  - (b) Give an example of a continuous function  $f: \mathbb{R} \longrightarrow \mathbb{R}$  which is not differentiable at x = 2. Prove that your function is not differentiable at x = 2. [25 Marks]

(c) State the Mean Value Theorem. Apply the Mean Value Theorem to the function  $e^x$  on [0, b], where 0 < b, to show that

$$b < e^b - 1 < be^b.$$

[40 Marks]

- Q6. (a) State the *Bolzano-Weierstrass Theorem* for sequence of real numbers.

  [15 Marks]
  - (b) Give formal definition of what it means for a sequence of real numbers to be a cauchy sequence. [15 Marks]
  - (c) Prove that a sequence of real numbers is convergent if and only if it is a cauchy sequence. [50 Marks]
  - (d) Let  $(x_n)$  be a sequence of real number defined by  $x_n = (1 + \frac{1}{2} + \dots + \frac{1}{n}) \quad \forall n \in \mathbb{N}$ . Show that  $(x_n)$  is not convergent. [20 Marks]

