EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE - 2005/2006 & 2006/2007

SECOND SEMESTER (Mar./ April., 2008)

MT 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

Proper & Repeat

Answer **all** questions Time : **Three** hours

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1. (a) State the necessary and sufficient condition for the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

to be **exact**. [10 marks]

Hence solve the following differential equation

$$(\sin x\, \cos y + e^{2x})dx + (\cos x\, \sin y + \tan y)dy = 0$$

[30 marks]

(b) Show that the solution of the general homogeneous equation of the first order and degree $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ is

$$\log x = \int \frac{dv}{f(v) - v} + C,$$

where $v = \frac{y}{x}$ and C is a constant. [20 marks]

Hence solve the differential equation

$$(x^2 - y^2)dx + 2xy\,dy = 0.$$

[40 marks]

2. (a) If $F(D) = \sum_{i=0}^{n} p_i D^i$, where $D = \dfrac{d}{dx}$ and p_i, $i = 1, 2, \ldots, n$ are constants with $p_0 \neq 0$, Prove the following formulas:

   i. $\dfrac{1}{F(D)}\, e^{\alpha x} = \dfrac{1}{F(\alpha)}\, e^{\alpha x}$, where α is a constant and $F(\alpha) \neq 0$;

   ii. $\dfrac{1}{F(D)}\, e^{\alpha x}\, V = e^{\alpha x} \dfrac{1}{F(D+\alpha)}\, V$, where V is a function of x.

[40 marks] .

(b) Find the general solution of the following differential equations by using the results in (a).

   i. $(D^3 + 4D^2 + 4D)y = 8e^{-2x}$.

   ii. $(D^3 - 3D^2 - 6D + 8)y = x\, e^{-3x}$.

[60 marks]

3. (a) Let $x + 1 = e^t$. Show that

$$(x+1)\frac{d}{dx} \equiv \mathcal{D},$$

and

$$(x+1)^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D}.$$

where $\mathcal{D} \equiv \dfrac{d}{dt}$.

[20 marks]

Use the above results to find the general solution of the following differential equation

$$[(x+1)^2 D^2 + (x+1)D - 1]y = \ln(x+1)^2 + x - 1.$$

[30 marks]

(b) With $D \equiv \frac{d}{dt}$, solve the following simultaneous differential equations

$$(5D+4)y - (2D+1)z = e^{-x},$$
$$(D+8)y - 3z = 5e^{-x}.$$

[50 marks]

4. Use the method of Frobenius to obtain two linearly independent solutions in series for the following differential equation

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 7y = 0.$$

[100 marks]

5. (a) Write down the condition of integrability of the total differential equation

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0.$$

[5 marks]

Hence solve the following equation

$$yz\log z\,dx - zx\log z\,dy + xy\,dz = 0$$

[15 marks]

(b) Solve the following system of differential equations:

i. $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$;

ii. $\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{4xy^2-2z}$.

[30 marks]

(c) Find the general solution of the following linear partial differential equations:

i. $z = px + qy + p^2 + pq + q^2$;

ii. $(y-z)p + (z-x)q = y - x$.

[30 marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$$

Here, $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$. [20 marks]

6. (a) Obtain Fourier series expansion of

$$f(x) = \begin{cases} 2x & \text{when} \quad 0 \leqslant x < 3, \\ 0 & \text{when} \quad -3 < x < 0 \end{cases}$$

[40 marks]

(b) Use the finite Fourier transformation to show the solution of the partial differential equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},$$

subject to the boundary condition:

$V(0,t) = 0, \quad V(4,t) = 0, \quad V(x,0) = 2x$, where $0 < x < 4,\ t > 0$

$$V(x,t) = \frac{-16}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{\frac{-n^2\pi^2 t}{16}} \cos n\pi \, \sin \frac{n\pi x}{4}.$$

[50 marks]

(c) Prove the following identities for Bessel function

i. $J_{-v}(x) = (-1)^v J_v(x), \quad v \geq 1;$

ii. $J'_v - \dfrac{v}{x} J_v(x) = -J_{v+1}(x).$ [10 marks]