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EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE - 2005/2006 & 2006/2007

(Mar./Apr.' 2008)

SECOND SEMESTER

ST 104 - DISTRIBUTION THEORY

(Proper and Repeat)

Answer all questions

Time : Three hours

Q1. (a) The trinomial distribution of two random variables X and Y is given by:

$$f_{X,Y}(x, y) = \frac{n!}{x!y!(n-x-y)!} p^x q^y (1-p-q)^{n-x-y}$$
$$\text{for } x, y = 0, 1, \cdots, n \text{ and } x + y \leq n,$$

where $0 \leq p$, $0 \leq q$ and $p + q \leq 1$.

(i) Find the marginal distribution of X and Y.

(ii) Find the conditional distributions of X and Y and obtain $E(Y/X = x)$ and $E(X/Y = y)$.

(iii) Find the correlation coefficient between X and Y.

(b) If $X_1, X_2, \cdots, X_k$ are k independent Poisson variates with parameters $\lambda_1, \lambda_2, \cdots, \lambda_k$ respectively, prove that the conditional distribution

$P(X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k/X)$, where $X = X_1 + X_2 + \cdots, X_k$ is fixed, is multinomial.

Q2. A particular fast food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and leaving the service window, and Y_2, the time that a customer waits in line before reaching the service window. Because Y_1 contains the time a customer waits in line, we must have $Y_1 \geq Y_2$. The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $P(Y_1 < 2, Y_2 > 1)$;

(b) Find $P(Y_1 \geq 2Y_2)$.

(c) If 2 minutes elapse between a customer's arrival at the store and departure from the service window, find the probability that he waited in line less than 1 minute to reach the window.

(d) Are Y_1 and Y_2 independent?

(e) The random variable $Y_1 - Y_2$ represents the time spent at the service window. Find $E(Y_1 - Y_2)$ and $V(Y_1 - Y_2)$. Is it highly likely that a customer would spend more than 2 minutes at the service window?

Q3. (a) Suppose that the length of time Y that it takes a worker to complete a certain task has the probability density function.

$$f(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta; \\ 0, & \text{elsewhere,} \end{cases}$$

where θ is a positive constant that represents the minimum time to task completion. Let $Y_1, Y_2, \cdots, Y_n$ denote a random sample of completion times from this distribution.

i. Find the density function for $Y_{(1)} = \min(Y_1, Y_2, \cdots, Y_n)$.

ii. Find $E(Y_{(1)})$.

(b) A bottling machine can be regulated so that it discharges an average of μ ounces per bottle. It has been observed that the amount of fill dispensed by the machine is normally distributed with $\sigma = 1.0$ ounce.

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i. A sample of $n = 9$ filled bottles is randomly selected from the output of the machine on a given day (all bottles is with the same machine setting)and the amount of fill measured for each. Find the probability that the sample mean, $\bar{Y}$ be within 0.3 ounce of the true mean μ for that particular setting.

ii. How many observations should be included in the sample if we wish $\bar{Y}$ to be within 0.3 ounce of μ with probability 0.95 ?

2. (a) Let X, Y be a two-dimensional non-negative continuous random variables having the joint density:

$$f(x,y) = \begin{cases} 4xy\, e^{-(x^2+y^2)}, & x \geq 0, y \geq 0; \\ 0, & \text{elsewhere.} \end{cases}$$

Prove the density function of $U = \sqrt{X^2 + Y^2}$ is

$$h(u) = \begin{cases} 2u^3\, e^{-u^2}, & 0 \leq u < \infty; \\ 0, & \text{elsewhere.} \end{cases}$$

(b) Two efficiency experts take independent measurements Y_1 and Y_2 on the length of time it takes workers to complete a certain task. Each measurement is assumed to have the density function given by

$$f(y) = \begin{cases} \frac{1}{4} y e^{-y/2}, & y > 0; \\ 0, & \text{elsewhere.} \end{cases}$$

Find the density function for the average $U = \frac{Y_1 + Y_2}{2}$.

(c) Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Find $E(S^2)$ and $V(S^2)$ where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

Q5. (a) If X has a Poisson distribution

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}, \quad r = 0, 1, 2 \ldots$$

where the parameter λ is a random variable of the continuous type with the density function

$$f(\lambda) = \frac{a^\nu e^{-a\lambda}\lambda^{\nu-1}}{\Gamma(\nu)}, \lambda \geq 0, a > 0, \nu > 0,$$

derive the distribution of X.

(b) Let $Y_1, Y_2, \cdots, Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Then show that $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is normally distributed with a mean of μ and a variance of $\frac{\sigma^2}{n}$.

(c) State the central limit theorem.

Q6. (a) Given the joint density function of X and Y as

$$f(x, y) = \begin{cases} \frac{1}{2} x e^{-y}, & 0 < x < 2, \ y > 0; \\ 0, & \text{elsewhere.} \end{cases}$$

Find the distribution of $X + Y$.

(b) The random variable X has the probability density function

$$\text{f(x)} = \begin{cases} 2e^{-2x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the moment generating function of X and hence find the mean and variance of X.

Show also that the median of the distribution is $\frac{1}{2}\ln 2$ and the inter-quartile range is $\frac{1}{2}\ln 3$.