EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE - (2003/2004)

(NOV./DEC.' 2004)

FIRST SEMESTER

MT 101 - FOUNDATION OF MATHEMATICS

Answer all questions — Time : Three hours

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1. (a) Let p, q and r be three propositions. Prove the following:

   i. $(p \wedge q) \vee > p \equiv\ > p \vee q$;

   ii. $(p \wedge q) \longrightarrow r \equiv (p \longrightarrow r) \vee (q \longrightarrow r)$;

   iii. $p \longrightarrow (q \longrightarrow r) \equiv (p \wedge > r) \longrightarrow > q$.

   iv. $[> p \wedge (> q \wedge r)] \vee (q \vee r) \vee (p \wedge r) \equiv q \vee r$.

   (b) Test the validity of the argument:

   Either Aruni is Anura's sister or Rani is not Runil's wife.

   Runil is Rani's husband or Aruni is not married.

   Anura is a bachelor if and only if Aruni is not married.

   Anura is married.

   ....

   Therefore, Anura is Aruni's brother.

2. What is meant by a set?

(a) Let A, B and C be subsets of a universal set E. Prove the following:

i. $(A \setminus B) \setminus C \subseteq A \setminus (B \setminus C)$;

ii. $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$;

iii. $A \times (B \triangle C) = (A \times B) \triangle (A \times C)$.

(b) At least 60% of a batch of students study Group Theory, at least 75% study Analysis, at least 80% study Statistics and at least 90% study Vector Calculus. What percentage (at least) must study all four subjects?

3. What is meant by an equivalence relation on a set?

(a) Let A be a set and let $\sim$ be an equivalence relation on A. Prove the following:

i. $[a] \neq \phi \ \forall\, a \in A$;

ii. $a \sim b \Leftrightarrow [a] = [b] \ \forall\, a,\, b \in A$;

iii. $b \in [a] \Leftrightarrow [a] = [b] \ \forall\, a,\, b \in A$;

iv. Either $[a] = [b]$ or $[a] \cap [b] = \phi \ \forall\, a,\, b \in A$.

(b) Let $S = \{(x,\, y) \in \mathbb{R}^2 \,/\, x \neq 0,\ y \neq 0\}$.

Define a relation ρ on S by
$(x,\, y)\, \rho\, (x_1,\, y_1) \Leftrightarrow (x\, y_1)^{2} = (y\, x_1)^{2}$ for any $(x,\, y),\, (x_1,\, y_1) \in S$.

Show that ρ is an equivalence relation.

Show also that $(x,\, y)\ \rho\ (x_1,\, y_1)$ if and only if there is a non-zero real number k such that $x = k\, x_1,\ y = \pm\, k\, y_1$. Sketch the ρ - class of the element $(1,\, 2)$.

4. (a) Define the following terms:

   i. Injective function;

   ii. Surjective function;

   iii. Bijective function;

   (b) $f : S \to T$ be a function and let A, B be subsets of S

   i. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.

   ii. Prove that $f(A \cup B) = f(A) \cup f(B)$.

   iii. Is it true that $f(A) \cap f(B) \subseteq f(A \cap B)$? Justify your answer.

   (c) If f, g and h are mapping from a set X to X such that $f \circ g = h \circ f = I_X$, the identity mapping from X to X, then show that f is a bijective and $f^{-1} = g = h$.

5. (a) Define the term "partially ordered set".

   State when a partially ordered set becomes totally ordered set.

   let A be a non-empty set and $P(A)$ a power set of A. Define a relation '$\preceq$' on $P(A)$ as $X \preceq Y$ if and only if X is a subset of Y.

   Prove that $(P(A), \preceq)$ is a partially ordered set.

   Prove also that $(P(A), \preceq)$ is not a totally ordered set if and only if $|A| > 1$.

   (b) Define the following elements of a partially ordered set .

   i. First element,

   ii. Last element,

   iii. Minimal element.

   Show that every partially ordered set has at most one first element and at most one last element.

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Show also that if a totally ordered set $(A, \preceq)$ has minimal element, then it will be the first element.

6. (a) Define the following:
   i. Greatest common divisor (gcd) of two integers,
   ii. A prime number.

   (b) Prove that any integer $n > 1$ can be expressed uniquely (except for order) as a product of primes.

   (c) Let $\gcd(a, b) = d$, where a and b are integers. Prove the following:
   i. $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$,
   ii. For any integer x, $\gcd(a, b) = \gcd(a, b + a\,x)$.

   (d) Prove that if $p\,|ab$, then $p|a$ or $p|b$, where p is a prime.

   (e) State necessary and sufficient condition for a linear Diaphantine equation $ax + by = c$ has a solution.

   Find the general solution (if there exists) of the following linear Diaphantine equations:

   i. $2x + 3y = 4$,

   ii. $10x - 8y = 3$.