

EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

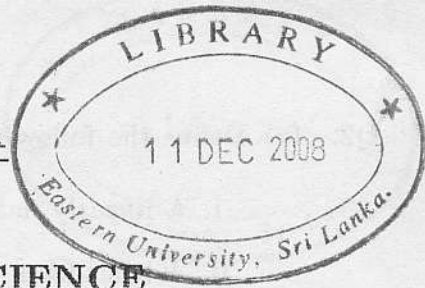
EXTERNAL DEGREE EXAMINATION IN SCIENCE

SECOND YEAR FIRST SEMESTER -(2003/2004) & (2004/2005)

(JULY/AUGUST' 2008)

EXTMT 201 - VECTOR SPACES AND MATRICES

PROPER AND REPEAT



Answer all questions

Time: Three hours

Q1. (a) Define what is meant by

(i) a vector space;

(ii) a subspace of a vector space.

Let  $V = \{P(x) = a_0x^2 + a_1x + a_2 : a_0, a_1, a_2, x \in \mathbb{R}\}$  be a set of all polynomials of degree  $\leq 2$ . Prove that  $V$  is a vector space over  $\mathbb{R}$  with the following operations:

$$(P + Q)(x) = P(x) + Q(x);$$

$$(\alpha P)(x) = \alpha P(x), \text{ for all } P(x), Q(x) \in V \text{ and for all } \alpha, x \in \mathbb{R}.$$

Is it true that the set of all polynomials of degree 2 forms a vector space?

Justify your answer.

(b) Let  $W_1$  and  $W_2$  be two subspaces of a vector space  $V$  over a field  $F$  and let  $A_1$  and  $A_2$  be non-empty subsets of  $V$ . Prove with the usual notations that

(i)  $W_1 + W_2 = \langle W_1 \cup W_2 \rangle$  ;

(ii) if  $\langle A_1 \rangle = W_1$  and  $\langle A_2 \rangle = W_2$  then  $\langle A_1 \cup A_2 \rangle = W_1 + W_2$ .

Q2. (a) Define the following:

- i. A linearly independent set of vectors;
- ii. A basis for a vector space;
- iii. Direct sum of two subspaces of a vector space.

(b) Let  $W_1, W_2$  be two subspaces of a vector space  $V$  over the field  $F$ . Prove that  $V$  is the direct sum of  $W_1$  and  $W_2$  if and only if each vector  $u \in V$  has unique representation  $u = w_1 + w_2$ , for some  $w_1 \in W_1$  and  $w_2 \in W_2$ .

Let  $W_1$  and  $W_2$  be two subspaces of  $\mathbb{R}^3$  defined by

$$W_1 = \{(a, b, c) : a = b = c, a, b, c \in \mathbb{R}\} \text{ and } W_2 = \{(0, x, y) : x, y \in \mathbb{R}\}.$$

Show that  $\mathbb{R}^3 = W_1 \oplus W_2$ .

- (c) i. Show that  $S = \{1, x, x^2\}$  is a basis of the set of all polynomials of degree  $\leq 2$ .
- ii. State Steinitz replacement theorem for a vector space .

Use this theorem to prove, for  $n$ -dimensional vector space  $V$  if  $\langle \{v_1, v_2, \dots, v_n\} \rangle = V$ , then  $\{v_1, v_2, \dots, v_n\}$  is a basis for  $V$ .

Q3. (a) Define

(i) Range space  $R(T)$ ;

(ii) Null space  $N(T)$

of a linear transformation  $T$  from a vector space  $V$  into another vector space  $W$ .

Find  $R(T)$ ,  $N(T)$  of the linear transformation  $T : V \rightarrow \mathbb{R}^2$ , defined by

$$T(a+bx+cx^2) = (a-b, b-c), \text{ where } V = \{a_0x^2 + a_1x + a_2 : a_0, a_1, a_2, x \in \mathbb{R}\}$$

Verify the equation  $\dim V = \dim(R(T)) + \dim(N(T))$  for this linear transformation.

(b) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear transformation defined by

$$T(x, y, z) = (x+2y, x+y+z, z), \text{ and let } B_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

and  $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$  be bases for  $\mathbb{R}^3$ .

Find

- (i) The matrix representation of  $T$  with respect to the basis  $B_1$ ;
- (ii) The matrix representation of  $T$  with respect to the basis  $B_2$  by using the transition matrix.

Q4. (a) Define the following terms:

- (i) Rank of a matrix;
- (ii) Echelon form of a matrix;
- (iii) Row reduced echelon form of a matrix.

(b) Let  $A$  be an  $m \times n$  matrix. Prove that

- (i) row rank of  $A$  is equal to column rank of  $A$ ;
- (ii) if  $B$  is an  $m \times n$  matrix obtained by performing an elementary row operation on  $A$ , then  $r(A) = r(B)$ .

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{pmatrix}$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 5 & 6 & 8 & -1 \\ 4 & 3 & 0 & 0 \\ 10 & 12 & 16 & -2 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$



Q5. (a) Define the term non-singular matrix.

Let  $P, Q$  and  $R$  be square matrices of the same order, where  $P$  and  $R$  are non-singular. Let  $O$  be the zero matrix of the same order. Prove that

the inverse of the block matrix  $\begin{pmatrix} P & O \\ \dots & \dots \\ Q & R \end{pmatrix}$  is

$$\begin{pmatrix} P^{-1} & O \\ \dots & \dots \\ -R^{-1}QP^{-1} & R^{-1} \end{pmatrix}.$$

Hence find the inverse of the matrix

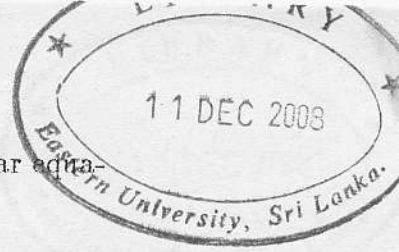
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(b) If  $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ , then use the mathematical induction to prove

$$A^n = \begin{pmatrix} 1 + 2n & -4n \\ n & 1 - 2n \end{pmatrix}, \text{ for all } n \in \mathbb{N}.$$

(c) Show that  $\det A = (a - b)(b - c)(c - a)(a + b + c)$  for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{pmatrix}, \text{ where } a, b, c \in \mathbb{R}.$$



6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Reduce the augmented matrix of the following system of linear equations

$$\lambda x + y + z = 1$$

$$x + \lambda y + z = 1$$

$$x + y + \lambda z = 1$$

to its row reduced echelon form and hence determine the values of  $\lambda$  such that the system has

- (i) a unique solution;
  - (ii) no solution;
  - (iii) more than one solution.
- (b) i. What is Cramer's rule?
- ii. Use Cramer's rule to solve the following system of linear equations.

$$2x_1 - 5x_2 + 2x_3 = 7$$

$$x_1 + 2x_2 - 4x_3 = 3$$

$$3x_1 - 4x_2 - 6x_3 = 5.$$