EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE

SECOND YEAR FIRST SEMESTER -(2003/2004)&(2004/2005)

(JULY/AUGUST' 2008)

EXTMT 201 - VECTOR SPACES AND MATRICES

PROPER AND REPEAT

Answer all questions **Time: Three hours**

Q1. (a) Define what is meant by

(i) a vector space;

(ii) a subspace of a vector space.

Let $V = \{P(x) = a_0x^2 + a_1x + a_2 : a_0, a_1, a_2, x \in \mathbb{R}\}$ be a set of all polynomials of degree ≤ 2. Prove that V is a vector space over $\mathbb{R}$ with the following operations:

$(P+Q)(x) = P(x) + Q(x)$;

$(\alpha P)(x) = \alpha P(x)$, for all $P(x), Q(x) \in V$ and for all $\alpha, x \in \mathbb{R}$.

Is it true that the set of all polynomials of degree 2 forms a vector space? Justify your answer.

(b) Let W_1 and W_2 be two subspaces of a vector space V over a field F and let A_1 and A_2 be non-empty subsets of V. Prove with the usual notations that

(i) $W_1 + W_2 = < W_1 \cup W_2 >$;

(ii) if $< A_1 > = W_1$ and $< A_2 > = W_2$ then $< A_1 \cup A_2 > = W_1 + W_2$.

Q2. (a) Define the following:

i. A linearly independent set of vectors;

ii. A basis for a vector space;

iii. Direct sum of two subspaces of a vector space.

(b) Let W_1, W_2 be two subspaces of a vector space V over the field F. Prove that V is the direct sum of W_1 and W_2 if and only if each vector $u \in V$ has unique representation $u = w_1 + w_2$, for some $w_1 \in W_1$ and $w_2 \in W_2$.

Let W_1 and W_2 be two subspaces of $\mathbb{R}^3$ defined by

$W_1 = \{(a, b, c) : a = b = c\ ,\ a, b, c \in \mathbb{R}\}$ and $W_2 = \{(0, x, y) : x, y \in \mathbb{R}\}$.

Show that $\mathbb{R}^3 = W_1 \oplus W_2$.

(c) i. Show that $S = \{1, x, x^2\}$ is a basis of the set of all polynomials of degree ≤ 2.

ii. State Stibnitz replacement theorem for a vector space .

Use this theorem to prove, for $n-$dimensional vector space V if $\langle\{v_1, v_2, \cdots, v_n\}\rangle = V$, then $\{v_1, v_2, \cdots, v_n\}$ is a basis for V.

Q3. (a) Define

(i) Range space $R(T)$;

(ii) Null space $N(T)$

of a linear transformation T from a vector space V into another vector space W.

Find $R(T)$, $N(T)$ of the linear transformation $T : V \to \mathbb{R}^2$, defined by

$T(a+bx+cx^2) = (a-b, b-c)$, where $V = \{a_0x^2 + a_1x + a_2 : a_0, a_1, a_2, x \in \mathbb{R}\}$

Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$T(x, y, z) = (x+2y,\ x+y+z,\ z)$, and let $B_1 = \{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$

and $B_2 = \{(1,1,0), (0,1,1), (1,0,1)\}$ be bases for $\mathbb{R}^3$.

Find

(i) The matrix representation of T with respect to the basis B_1;

(ii) The matrix representation of T with respect to the basis B_2 by using the transition matrix.

Q4. (a) Define the following terms:

(i) Rank of a matrix;

(ii) Echelon form of a matrix;

(iii) Row reduced echelon form of a matrix.

(b) Let A be an $m \times n$ matrix. Prove that

(i) row rank of A is equal to column rank of A;

(ii) if B is an $m \times n$ matrix obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 5 & 6 & 8 & -1 \\ 4 & 3 & 0 & 0 \\ 10 & 12 & 16 & -2 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$

Q5. (a) Define the term non-singular matrix.

Let P, Q and R be square matrices of the same order, where P and R are non-singular. Let O be the zero matrix of the same order. Prove that the inverse of the block matrix $\left(\begin{array}{c|c} P & O \\ \cdots & \cdots \\ Q & R \end{array}\right)$ is

$$\left(\begin{array}{c|c} P^{-1} & O \\ \cdots\cdots\cdots & \cdots \\ -R^{-1}QP^{-1} & R^{-1} \end{array}\right).$$

Hence find the inverse of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(b) If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, then use the mathematical induction to prove

$A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$, for all $n \in \mathbb{N}$.

(c) Show that $det A = (a-b)(b-c)(c-a)(a+b+c)$ for

$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{pmatrix}$, where $a, b, c \in \mathbb{R}$.

6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Reduce the augmented matrix of the following system of linear equations

$$\lambda x + y + z = 1$$

$$x + \lambda y + z = 1$$

$$x + y + \lambda z = 1$$

to its row reduced echelon form and hence determine the values of λ such that the system has

(i) a unique solution;

(ii) no solution;

(iii) more than one solution.

(b) i. What is Crammer's rule?

ii. Use Crammer's rule to solve the following system of linear equations.

$$2x_1 - 5x_2 + 2x_3 = 7$$

$$x_1 + 2x_2 - 4x_3 = 3$$

$$3x_1 - 4x_2 - 6x_3 = 5.$$