EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2005/06)

SECOND SEMESTER (March/April '2008)

MT 301 - GROUP THEORY

(Proper and Repeat)

Answer all questions **Time: Three hours**

1. (a) Define the following terms:

   i. group,

   ii. cyclic group,

   iii. abelian group.

   Prove that every subgroup of a cyclic group is cyclic.

   Is the converse part true? Justify your answer.

   (b) State and prove Lagrange's theorem.

   Let G be a group with sub groups H and K of orders 69 and 75.

   Prove that $H \cap K$ is cyclic.

   (Any result used should be proved)

   (c) i. Let G be a group with a subgroup H. Show that if $g \in G$ satisfies $g^2 \in H$ but $g \notin H$, then $g^7 \notin H$ and $g^{-3} \notin H$.

   ii. Let $G(\subseteq \mathbb{Z})$ with a binary operation $*$, which is defined by $x * y = 2x + y$. Is $(G, *)$ a group? Justify your answer.

2. (a) Define the following:

   i. normal subgroup of a group,

   ii. homomorphism.

   (b) Prove the following:

   i. if $H \trianglelefteq G$ and $K \trianglelefteq G$ then $HK \trianglelefteq G$,

   ii. let $\phi : G \to G_1$ be a homomorphism. If $H \leq G$ then $\phi(H) \leq G_1$.

   What condition ϕ should satisfy in order that $\phi(H) \trianglelefteq G_1$ when $H \trianglelefteq G$? Prove this result, when ϕ satisfies this condition.

   (c) Let $Z(G) = \{x \in G \mid xg = gx, \forall\, g \in G\}$. Prove the following:

   i. $Z(G) = \bigcap_{a \in G} C(a)$, where $C(a) = \{g \in G \mid ga = ag\}$,

   ii. $Z(G) \trianglelefteq G$,

   iii. if $G/Z(G)$ is cyclic, then G is abelian.

3. (a) State the first isomorphism theorem.

   Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove the following:

   i. $K \trianglelefteq H$,

   ii. $H/K \trianglelefteq G/K$,

   iii. $\dfrac{H/K}{G/K} \cong G/H$.

   (b) Write down the class equation of a finite group G.

   Let G be a group of order p^n, where p is a prime number. Prove the following:

   i. $Z(G)$ is non-trivial,

   ii. if $n = 2$ then $Z(G) = G$.

   (State any result that you may use)

4. (a) Define commutator subgroup G' of a group G.

Prove the following:

i. G is abelian if and only if $G' = \{e\}$,

ii. $G' \trianglelefteq G$,

iii. G/G' is abelian.

(b) Let $H \trianglelefteq G, P = \{K \leq G \mid H \subseteq K\}$ and $Q = \{K' \mid K' \leq G/H\}$.

Prove that there exists a one to one correspondence between P and Q.

5. (a) What is meant by the "internal direct product" as applied to a group.

Is it true that all the groups satisfy the internal direct product property? Justify your answer.

Let H and K be two subgroups of a group G, prove that G is a direct product of H and K if and only if

i. each $x \in G$ can be uniquely expressed in the form
$x = hk$, where $h \in H, k \in K$,

ii. $hk = kh$ for any $h \in H, k \in K$.

(b) Define the term "p-group".

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

6. (a) Define the following terms as applied to a permutation group:

   i. cycle of order r,

   ii. transposition,

   iii. signature.

(b) Prove that the permutation group on n symbols (S_n) is a finite group of order $n!$.

(c) Prove that every permutation in S_n can be expressed as a product of transpositions. Hence prove an even permutation can be expressed as a product of even number of transpositions.

Write down the following permutation in S_9 as a product of transpositions

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 7 & 9 & 8 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$$

(d) Prove that the set of all even permutations A_n forms a normal subgroup of S_n. Hence prove S_n/A_n is a cyclic group of order 2.