# EASTERN UNIVERSITY, SRI LANKA

## THIRD EXAMINATION IN SCIENCE 2005/2006

## March/April' 2008

## SECOND SEMESTER

## MT 303 - FUNCTIONAL ANALYSIS

## Proper & Repeat

Answer all questions　　　　Time:Two hours

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Q1. Define the following:

(i) Banach Space;

(ii) Separable normed linear space.

(a) Let $\ell^1 = \{\ \mathbf{x} = (x_1, x_2, ...) \mid x_i \in K,\ \sum_{i=1}^{\infty} |x_i| < \infty\}$ and let $||\mathbf{x}|| = \sum_{i=1}^{\infty} |x_i|$.

Prove that $\ell^1$ is a Banach space here $K$ is a field.

(b) Show that, $\ell^2 = \{\ \mathbf{x} = (x_1, x_2, ...) \mid x_i \in K,\ \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with norm

$$||\mathbf{x}|| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$$ is separable.

(c) Show that $\ell^\infty = \{\ \mathbf{x} = (x_1, x_2, ....) \mid x_i \in K,\ \sup_i |x_i| < \infty\}$ with the usual norm is non separable.

Q2. (a) State and prove the Riesz's lemma.

(b) Prove that a normed linear space $X$ is finite dimensional if and only if the closed unit ball $\{\ \mathbf{x} \mid ||\mathbf{x}|| \leq 1\ \}$ is compact.

(c) Show that two norms on a linear space are equivalent if and only if every Cauchy sequence with respect to one of the norms is a Cauchy sequence with respect to other norm.

Q3. (a) Let $T$ be a linear operator from a normed linear space $X$ into a normed linea space $Y$. Prove that $T$ is continuous if and only if $T$ is bounded.

(b) Let $T$ be a linear operator from a normed linear space $X$ into a normed linea space $Y$. Show that the null space $N(T)$ is closed.

(c) Let $T$ be a linear operator from a normed linear space $X$ into a normed linea space $Y$. Show that $T$ is bounded if and only if $T$ maps bounded sets in $X$ into bounded sets in $Y$.

(d) Define the norm of a bounded linear operator between two normed linea spaces. Let $T_1 : Y \longrightarrow Z, T_2 : X \longrightarrow Y$ and $T : X \longrightarrow X$ be bounded linear operators, where $X, Y$ and $Z$ are normed linear spaces. Show that $||T_1T_2|| \leq ||T_1|| \, ||T_2||$ and $||T^n|| \leq ||T||^n$ for all $n \in \mathbb{N}$.

Q4. State the Hahn Banach theorem for real normed linear space.

(a) Let $\mathbf{x}_0$ be any non zero element in a normed linear space X. Prove that there exists $g \in X^*$ such that $||g||_{X^*} = 1$ and $g(\mathbf{x}_0) = ||\mathbf{x}_0||$.

(b) Let $M$ be a subspace of a normed linear space $X$, $\mathbf{x}_0 \notin M$ and if $\delta = \inf_{\mathbf{x} \in M} ||\mathbf{x} - \mathbf{x}_0|| > 0$, then prove that there exists $g \in X^*$ such that $g(\mathbf{x}) = 0, \quad \forall\, \mathbf{x} \in M, \quad g(\mathbf{x}_0) = \delta$ and $||g||_{X^*} = 1$.

(c) Let $X$ be a normed linear space. If $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $\mathbf{x}_1 \neq \mathbf{x}_2$, show that there exists $f \in X^*$ such that $f(\mathbf{x}_1) \neq f(\mathbf{x}_2)$.