EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE 2002/2003

(Apr./May ' 2004)

(Repeat)

MT 103 & 104 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

Answer four questions only selecting two questions from each section

Time : Two hours

Section A

1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\ \underline{b} - (\underline{a} \cdot \underline{b})\ \underline{c}\ .$$

Hence prove that

$$(\underline{a} \wedge \underline{b}) \cdot (\underline{c} \wedge \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})\ .$$

(b) Let $\underline{l}$, $\underline{m}$ and $\underline{n}$ be three non-zero and non-coplanar vectors such that any two of them are not parallel. By Considering the vector product $(\underline{r} \wedge \underline{l}) \wedge (\underline{m} \wedge \underline{n})$, prove that any vector $\underline{r}$ can be expressed in the form

$$\underline{r} = (\underline{r} \cdot \underline{\alpha})\ \underline{l} + (\underline{r} \cdot \underline{\beta})\ \underline{m} + (\underline{r} \cdot \underline{\gamma})\ \underline{n}\ .$$

Find the vectors $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ in terms of $\underline{l}$, $\underline{m}$, $\underline{n}$.

(c) A vector $\underline{r}$ satisfies the equation

$$\underline{r} \wedge \underline{b} = \underline{c} \wedge \underline{b} \text{ and } \underline{r} \cdot \underline{a} = 0 ,$$

where $\underline{a}$ and $\underline{b}$ are non-zero and not perpendicular vectors. Show that $\underline{r}$ can be expressed in the form

$$\underline{r} = \underline{c} - \lambda \underline{b} ,$$

where λ is a scalar.

2. (a) Define the following terms:

   i. The gradient of a scalar field ϕ ,

   ii. The divergence of a vector field $\underline{F}$,

   iii. The curl of a vector field $\underline{F}$.

   (b) Prove the following:

   i. $\text{div}(\phi \underline{F}) = \phi \, \text{div}\underline{F} + \text{grad}\phi \cdot \underline{F}$;

   ii. $\text{curl}(\phi \underline{F}) = \phi \, \text{curl}\underline{F} + \text{grad}\phi \wedge \underline{F}$.

   (c) Let $\underline{a}$ be non-zero constant vector and $\underline{r}$ be a positive vector of a point such that $\underline{a} \cdot \underline{r} \neq 0$ and let n be a constant. If $\phi = (\underline{a} \cdot \underline{r})^n$, show that $\nabla^2 \phi = 0$ if and only if $n = 0$ or $n = 1$.

   If $r = |\, \underline{r} \,|$, find $\text{div}(r^n \underline{r})$ and $\underline{\nabla} \left(\frac{\underline{a} \cdot \underline{r}}{r^5} \right)$.

   Hence show that

$$\text{curl} \left[\left(\frac{\underline{a} \cdot \underline{r}}{r^5} \right) \underline{r} \right] = \frac{\underline{a} \wedge \underline{r}}{r^5} .$$

3. (a) State and prove Green's theorem on the plane.

   Verify Green's theorem in the plane for

$$\oint_C (xy + y^2) dx + x^2 dy,$$

where C is the closed boundary of the region defined by $y = x,\ y = x^2$.

(b) State the divergence theorem and use it to evaluate $\iint_S \underline{A} \cdot \underline{n}\, dS$, where $A = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and S is the surface of the cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1$ and $z = 0,\ z = 1$.

Section B

4. Prove that the radial and transverse component of the acceleration of a particle in a plane in terms of polar co-ordinates (r, θ) are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

respectively.

A particle of mass m rest on a smooth horizontal table attached to a fixed point on the table by a light elastic string of modules mg and unstretched length $'a'$. Initially the string is just taut and the particle is projected along the table in a direction perpendicular to the line of the string with velocity $\sqrt{\frac{4ga}{3}}$. Prove that if r is the distance of the particle from the fixed point at time t, then

$$\frac{d^2r}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g}{a}(r - a).$$

Prove also that the string will extend until its length is $2a$ and that the velocity of the particle is then half of its initial value.

5. A particle moves in a plane with velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body, where k is a constant. Neglecting the air resistance to the motion of the body, prove that the velocity of the body when it's path is inclined an angle ψ to the horizontal is

$$\frac{v_0 \sec \psi}{(\sec \psi + \tan \psi)^k}.$$

Prove that if $k = 1$ the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

6. Establish the equation

$$\underline{F}(t) = m(t)\frac{d\underline{v}}{dt} + \underline{v}_0\frac{dm(t)}{dt}$$

for the motion of a rocket of varying mass $m(t)$ moving in a straight line with velocity $\underline{v}$ under a force $\underline{F}(t)$, matter being emitted at a constant rate with a velocity $\underline{v}_0$ relative to the rocket.

A rocket is fired upwards. Matter is ejected with constant relative velocity gT, at a constant rate $\frac{2M}{T}$. Initially the mass of the rocket is $2M$, half of this is available for ejection. Neglecting the air resistance and variation in gravitational attraction, show that the greatest speed of the rocket is attained when the mass of the rocket is reduced to M, and this speed is

$$gT\left(\ln 2 - \frac{1}{2}\right).$$

Show also that the rocket will reach the greatest height given by

$$\frac{1}{2}gT^2(1 - \ln 2)^2.$$