EASTERN UNIVERSITY, SRI LANKA

FIRST YEAR EXAMINATION IN SCIENCE

2002/2003 & 2002/2003 (A)

SECOND SEMESTER

(April/May '2004)

MT 104 - DIFFERENTIAL EQUATIONS

AND

FOURIER SERIES

Answer **All** Questions   Time Allowed: **3 Hours**

Q1. (a) State the necessary and sufficient condition for the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

to be **exact**.   [10 Marks]

Hence solve the following differential equation

$$\left(2xy\,e^{x^2y} + y^2\,e^{xy^2} + 1\right)dx + \left(x^2\,e^{x^2y} + 2xy\,e^{xy^2} - 2y\right)dy = 0.$$

[20 Marks]

(b) By differentiating the equation

$$\int \frac{f(xy) + F(xy)}{f(xy) - F(xy)}\,\frac{d(xy)}{xy} + \log\frac{x}{y} = C,$$

where C is a constant, verify that $\dfrac{1}{xy\{f(xy) - F(xy)\}}$ is an integrating factor of

$$yf(xy)\,dx + xF(xy)\,dy = 0.$$

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Hence solve the differential equation

$$y(2xy + 1)\, dx + x(1 + 2xy - x^3y^3)\, dy = 0.$$

[20 Marks]

(c) If $\tan x$ is a particular solution of the following non-linear Riccati differential equation

$$\frac{dy}{dx} = 1 + y^2,$$

then obtain the general solution of the differential equation .

[30 Marks]

Q2. (a) If $F(D) = \displaystyle\sum_{i=0}^{n} p_i D^i$, where $D \equiv \dfrac{d}{dx}$ and p_i, $i = 1, \ldots, n$, are constants with $p_0 \neq 0$, then prove the following formulas:

(i) $\dfrac{1}{F(D)} e^{\alpha x} = \dfrac{1}{F(\alpha)} e^{\alpha x}$, α is a constant and $F(\alpha) \neq 0$;

(ii) $\dfrac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \dfrac{1}{F(D+\alpha)} V$, where V is a function only of x.

[40 Marks]

(b) Find the general solution of the following differential equations by using the results in (a).

(i) $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x}$.

(ii) $(D^3 - 3D^2 - 6D + 8)y = xe^{-3x}$.

[60 Marks]

Q3. (a) If $x = e^t$, then show that

$$x \frac{d}{dx} \equiv \mathcal{D}, \quad x^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D},$$

and

$$x^3 \frac{d^3}{dx^3} \equiv \mathcal{D}(\mathcal{D} - 1)(\mathcal{D} - 2),$$

where $D \equiv \dfrac{d}{dt}$.

[20 Marks]

Use the above results to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12 \log x}{x^2}.$$

[30 Marks]

(b) With $D \equiv \dfrac{d}{dt}$, solve the following simultaneous differential equations

$$D^2x - m^2y = 0,$$
$$D^2y + m^2x = 0.$$

[50 Marks]

Q4. Use the method of Frobenius to find the general solution of

$$(x-1)^2 \frac{d^2y}{dx^2} + (3x^2 - 4x + 1)\frac{dy}{dx} - 2y = 0$$

by expanding about $x = 1$.

[100 Marks]

Q5. (a) Solve the following system of differential equations:

(i) $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)};$

(ii) $\dfrac{dx}{x(2y^4 - z^4)} = \dfrac{dy}{y(z^4 - 2x^4)} = \dfrac{dz}{z(x^4 - y^4)}.$

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz = 0.$$

[5 Marks]

Hence solve the following equation

$$yz \log z\, dx - zx \log z\, dy + xy\, dz = 0.$$

[15 Marks]

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(c) Find the general solution of the following linear first-order partial differential equations:

   (i) $(y - z)p + (z - x)q = y - x$;

   (ii) $(x^2 + y^2 - yz)p - (x^2 + y^2 - xz)q = z(x - y)$.

[30 Marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$2xz - px^2 - 2qxy + pq = 0.$$

Here, $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

[20 Marks]

Q6. (a) Prove that if $-\pi \leq x \leq \pi$ and a is not an integer, then

$$\cos ax = \frac{2a \sin a\pi}{\pi} \left\{ \frac{1}{2a^2} - \frac{\cos x}{a^2 - 1} + \frac{\cos 2x}{a^2 - 4} - \cdots \right\}.$$

[20 Marks]

Use the above result to show that

$$\frac{a\pi}{\sin a\pi} = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n a^2}{a^2 - n^2}.$$

[20 Marks]

(b) Use Fourier transform to solve Dirichlet problem, that is, $U(x, y)$ satisfies the two-dimensional form of Laplace's equation
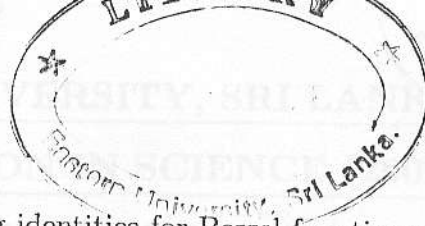
$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad -\infty < x < \infty, \ y \geq 0,$$

subject to the boundary conditions

$$U(x, 0) = f(x), \quad -\infty < x < \infty,$$
$$U(x, 0) \to 0 \text{ as } \sqrt{x^2 + y^2} \to \infty.$$

[50 Marks]

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(c) Prove the following identities for Bessel functions:

(i) $J_{-n}(x) = (-1)^n J_n(x), \quad \nu = n \geq 1$;

(ii) $J_\nu' - \dfrac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x)$.

[10 Marks]