EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 2005/2006

FIRST SEMESTER(August/September, 2007)

MT 201 - VECTOR SPACES AND MATRICES

(PROPER AND REPEAT)

Answer all Questions **Time: Three hours**

01. (a) Explain what is meant by a vector space?

Let $V = \{x \mid x \in \mathbb{R}, x > 0\}$ and let the vector addition and scalar multiplication be defined in the usual way of addition and scalar multiplication over the field $\mathbb{R}$. Is V a vector space over $\mathbb{R}$? Justify your answer. [20 marks]

(b) State the necessary and sufficient condition for a non-empty subset to be a subspace of a vector space.

Let S be a non empty subset of a vector space V over the field $\mathbb{F}$. Prove that

(i) $\langle S \rangle$ is the set of all linear combination of the element in S.

(ii) the intersection of all the subspaces containing S is the smallest subspace containing S. [45 marks]

(c) (i) Prove that $\langle x, y, z \rangle = \langle x + y, x + z, y + z \rangle$, for $x, y, z \in \mathbb{R}$.

(ii) Define the term **"direct sum"** of two subspaces of a vector space.

Let V be the vector space of n-square matrices over the field $\mathbb{R}$. If U and W are two subspaces of symmetric and anti-symmetric matrices respectively, then show that $V = U \oplus W$. [35 marks]

02. (a) What is meant by the following in a vector space:

(i) Linearly independent vectors;

(ii) Bases;

(iii) Dimension.

Find the values of the real number λ so that the following vectors are linearly independent in $\mathbb{R}^3$?

$$X_1 = \begin{pmatrix} \lambda \\ -1 \\ -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 \\ \lambda \\ -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 \\ -1 \\ \lambda \end{pmatrix}.$$

[30 marks]

(b) State and prove the Stibnitz representation theorem.

Hence deduce that if V is an n-dimensional vector space, then $(n+1)$ or more vectors of V form a linearly dependent set. [45 marks]

(State any results that you may use with out proof)

(c) State the dimension theorem and use it to show that

$$\dim(S \cap T) \geq 2,$$

where S and T are 6-dimensional subspaces of the vector space $\mathbb{F}^{10}$. In general, $\mathbb{F}^n$ denotes the set of all n-tuples of elements of the field $\mathbb{F}$. [25 marks]

03. Prove or disprove the following:

(a) $S = \{(x, y) \mid xy = 0, x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$. [10 marks]

(b) The functions that are linearly independent on an interval implies that they are linearly independent on a sub interval. [20 marks]

(c) Any linearly independent subset of a finite dimensional vector space can be extended to a basis. [30 marks]

(d) If $A = [a_{ij}]_{n \times n}$, then $A(adj A) = (det A)I_n$, where I_n is the identity matrix of order n. [20 marks]

(e) If $\phi : V \mapsto W$ is a linear transformation and $\text{Ker}(\phi) = \{0\}$, then ϕ is injective.

[20 marks]

04. (a) Define the following terms as applied to a linear transformation $\phi : V \mapsto W$, where V and W are vector spaces:

i. Range space $R(\phi)$.

ii. Null space $N(\phi)$.

Let ϕ be a linear transformation of $\mathbb{R}^3$ given by

$$\phi(x, y, z) = (x_1 + x_2 + 2x_3, 2x_1 + x_2 + x_3, 3x_1 - x_2 - 6x_3).$$

By finding the bases of $R(\phi)$ and $N(\phi)$, verify that

$$\dim(\mathbb{R}^3) = r(\phi) + n(\phi),$$

where $r(\phi)$ and $n(\phi)$ are, respectively, the rank and nullity of the linear transformation ϕ. [35 marks]

(b) What is meant by the following in a finite dimensional vector space:

i. coordinate vectors of a vector;

ii. transition matrix.

Prove that if A is the transition matrix from the basis $\{x_1, x_2, \cdots, x_n\}$ to the bases $\{y_1, y_2, \ldots, y_n\}$ of a vector space V and B is the transition matrix from the basis $\{y_1, y_2, \ldots, y_n\}$ to the basis $\{x_1, x_2, \ldots, x_n\}$, then $AB = BA = I$. [40 marks]

(c) If $B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $B_2 = \{(-2,1,0), (1,1,1), (-3,0,4)\}$ are two bases for the vector space $\mathbb{R}^3$, then verify the above result in (b). [25 marks]

05. (a) Let

$$A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}.$$

Using mathematical induction, show that

$$A^n = \frac{(3^n - 1)}{2} A + \frac{(3 - 3^n)}{2} I_2, \quad \text{for } n \geq 1.$$

Hence show that if a sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ satisfies the recurrence relation

$$x_{n+1} = 4x_n - 3x_{n-1}, \quad n \geq 1,$$

then find a formula for x_n in terms of x_1 and x_0. [30 marks]

(b) Define the following terms:

(i) Non-singular matrix;

(ii) Elementary row matrices;

(iii) Reduced row-echelon form of a matrix. [15 marks]

(c) Let A be a non-singular $n \times n$ matrix. Show that

(i) A is row-equivalent to identity matrix of order n.

(ii) A is a product of elementary row matrices. [25 marks]

(d) Using the notion of the partitioned matrix, show that

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

is non-singular, find A^{-1} and express A as a product of elementary row matrices. Also, using the equation

$$A^{-1} = \frac{1}{|A|} (\text{adj } A),$$

find A^{-1} and check with A^{-1} that you have obtained above. [30 marks]

06. (a) If $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and $(\beta_1, \beta_2, \dots, \beta_n)$ are solution of a system of linear equations,

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \le i \le m,$$

prove that

$$((1-t)\alpha_1 + t\beta_1), ((1-t)\alpha_2 + t\beta_2), \dots, ((1-t)\alpha_n + t\beta_n)$$

is also a solution of the above system of linear equations. [30 marks]

(b) Consider the following system of linear equations,

$$\begin{aligned} x - 2y + 3z &= 4, \\ 2x - 3y + az &= 5, \\ 3x - 4y + 5z &= b. \end{aligned}$$

Find the values of a and b such that the system of equations have

i. no solution,

ii. a unique solution,

iii. infinitely many solutions. [70 marks]