EASTERN UNIVERSITY, SRI LANKA

FIRST YEAR EXAMINATION IN SCIENCE

2003/2004

SECOND SEMESTER

(June/July - 2005)

Proper & Repeat

MT 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

Answer **All** Questions | Time Allowed: **3** Hours

Q1. (a) State the necessary and sufficient condition for the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

to be exact. [10 Marks]

Hence solve the following differential equation

$$\left(x\sqrt{x^2+y^2} - y\right)dx + \left(y\sqrt{x^2+y^2} - x\right)dy = 0.$$

[20 Marks]

(b) By differentiating the equation

$$\int \frac{f(xy) + F(xy)}{f(xy) - F(xy)} \frac{d(xy)}{xy} + \log\frac{x}{y} = C,$$

where C is a constant, verify that $\dfrac{1}{xy\{f(xy) - F(xy)\}}$ is an integrating factor of the differential equation

$$yf(xy)\,dx + xF(xy)\,dy = 0.$$

[20 Marks]

Hence solve the differential equation

$$y(1+2xy)\,dx + x(1-xy)\,dy = 0.$$

[20 Marks]

(c) If $y_1 = 2x$ is a particular solution of the following non-linear Riccati differential equation

$$\frac{dy}{dx} = 2 - 2xy + y^2,$$

obtain the general solution of the differential equation .

[30 Marks]

Q2. (a) If $F(D) = \sum_{i=0}^{n} p_i D^i$, where $D \equiv \frac{d}{dx}$ and p_i, $i = 1, \ldots, n$, are constants with $p_0 \neq 0$, prove the following formulas:

(i) $\frac{1}{F(D)}e^{\alpha x} = \frac{1}{F(\alpha)}e^{\alpha x}$, where α is a constant and $F(\alpha) \neq 0$;

(ii) $\frac{1}{F(D)}e^{\alpha x}V = e^{\alpha x}\frac{1}{F(D+\alpha)}V$, where V is a function of x.

[40 Marks]

(b) Find the general solution of the following differential equations by using the results in (a).

(i) $(D^4 - 2D^2 + 1)y = 40\cosh x$.

(ii) $(D^3 - 3D^2 - 6D + 8)y = xe^{-3x}$.

[60 Marks]

Q3. (a) Let $x = e^t$. Show that

$$x\frac{d}{dx} \equiv \mathcal{D}, \quad x^2\frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D},$$

and

$$x^3\frac{d^3}{dx^3} \equiv \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2),$$

where $\mathcal{D} \equiv \dfrac{d}{dt}$. [20 Marks]

Use the above results to find the general solution of the following differential equation

$$x^3y''' + xy' - y = 3x^4, \quad \text{where } ' \equiv \frac{d}{dx}.$$

[30 Marks]

(b) With $D \equiv \dfrac{d}{dt}$, solve the following simultaneous differential equations

$$\begin{aligned} D^2x - \alpha^2 y &= 0, \\ D^2y + \alpha^2 x &= 0. \end{aligned}$$

[50 Marks]

Q4. Use the method of Frobenius to find the general solution of

$$(x-1)^2\frac{d^2y}{dx^2} + (3x^2 - 4x + 1)\frac{dy}{dx} - 2y = 0$$

by expanding about $x = 1$. [100 Marks]

Q5. (a) Solve the following system of differential equations:

(i) $\dfrac{dx}{x(y^2-z^2)} = \dfrac{dy}{y(z^2-x^2)} = \dfrac{dz}{z(x^2-y^2)}$;

(ii) $\dfrac{dx}{x(y^3-2x^3)} = \dfrac{dy}{y(2y^3-x^3)} = \dfrac{dz}{9z(x^3-y^3)}$.

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz = 0.$$

[5 Marks]

Hence solve the following equation

$$(yz + xyz)\, dx + (zx + xyz)\, dy + (xy + xyz)\, dz = 0.$$

[15 Marks]

(c) Find the general solution of the following linear first-order partial differential equations:

(i) $(y - z)p + (z - x)q = y - x$;

(ii) $(x^2 + y^2 - yz)p - (x^2 + y^2 - xz)q = z(x - y)$;

where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

[30 Marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$p(q^2 + 1) + (b - z)q = 0.$$

Here, $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

[20 Marks]

Q6. (a) Prove that if $-\pi \leq x \leq \pi$ and a is not an integer, then

$$\cos ax = \frac{2a \sin a\pi}{\pi} \left\{ \frac{1}{2a^2} - \frac{\cos x}{a^2 - 1} + \frac{\cos 2x}{a^2 - 4} - \dots \right\}.$$

[20 Marks]

Use the above result to show that

$$\frac{a\pi}{\sin a\pi} = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n a^2}{a^2 - n^2}.$$

[20 Marks]

(b) Use Fourier transform to solve the one-dimensional heat equation

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2} = 0,$$

subject to the boundary conditions

$$U(0,t) = 0, \; U(x,0) = e^{-x}, \quad x > 0$$

and U(x,t) is bounded where $x > 0$ and $t > 0$.

[50 Marks]

(c) Prove the following identities for Bessel functions:

(i) $J_{-\nu}(x) = (-1)^{\nu} J_{\nu}(x), \;\; \nu \geq 1;$

(ii) $J_{\nu}' - \frac{\nu}{x} J_{\nu}(x) = -J_{\nu+1}(x).$

[10 Marks]