EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE - 2003/2004

(June/July'2005)

SECOND SEMESTER

ST 104 - DISTRIBUTION THEORY

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wer all questions | Time : Three hours

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1. If X and Y are two random variables having joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & if\ 0 < x < 2, 2 < y < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find

(a) marginal densities of X and Y.

(b) joint cumulative distribution function.

(c) $P(X < 1, Y < 3)$.

(d) $P(X + Y < 3)$.

(e) $P(X < 1 | Y < 3)$.

2. (a) A particular fast-food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and leaving the service window, and Y_2, the time that a customer waits in line before reaching the service window. The relative frequency distribution of observed values of Y_1 and Y_2 is modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}; & 0 \le y_2 \le y_1 < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

The random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window.

i. Find the probability density function for U.

ii. Find $E(U)$ and $V(U)$

(b) If X is a random variable with mean μ and variance σ^2, then for any positive number k, prove that

$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$.

3. (a) Suppose that the length of time Y that takes a worker to complete a certain task, has the probability density function

$$f(y) = \begin{cases} e^{-(y-\theta)}; & \text{y} > \theta, \\ 0 & \text{otherwise.} \end{cases}$$

where θ is a positive constant that represents the minimum time to task completion. Let $Y_1, Y_2, ..., Y_n$ denote a random sample of completion times from this distribution.

i. Find the density function for $Y_{(1)} = min(Y_1, ..., Y_n)$.

ii. Find $E(Y_{(1)})$.

(b) Let X be a standard normal variate. Show that $Y = X^2$ is a chi-square random variable with degrees of freedom 1.

(c) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a normal distribution with a mean μ and a variance of σ^2. If $Z_i = \dfrac{(Y_i - \mu)}{\sigma}$, show that $\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left[\dfrac{(Y_i - \mu)}{\sigma}\right]^2$ is a χ^2 distribution with n degrees of freedom.

4. (a) The joint density function of X and Y is given by

$$f_{XY}(x, y) = \frac{1}{2\Pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \right.\right.$$

where $x, y, \mu_1, \mu_2 \in \mathbb{R}, \sigma_1 > 0, \sigma_2 > 0$ and $| \rho | \leq 1$

i. Find the marginal density function of X. Name the density function.

ii. Find the conditional density function of Y given $X = x$. Name the density function.

iii. From (ii) deduce $E(Y|X = x)$.

A bottling machine can be regulated so that it discharges an average of μ ounces per bottle. It has been observed that the amount of fill dispensed by the machine is normally distributed with $\sigma = 1.0$ ounce. A sample of $n = 9$ filled bottles is randomly selected from the output of the machine on a given day (all bottles are with the same machine setting) and the ounce of fill measured for each. Find the probability that the sample mean differ from the true mean within 0.3 ounce for that particular setting.

pose n balls are randomly selected from a box containing W white and B black balls. Let enote the number of white balls selected and

$$X_i = \begin{cases} 1 & \textit{if the i}^{th} \textit{ ball selected is white,} \\ 0 & \text{otherwise.} \end{cases}$$

) Assume that the n balls are selected with replacement, that is a selected ball is put back in the box before the next ball is drawn. Find $E(X)$ and $V(X)$.

Assume that the n balls are selected without replacement, that is no selected balls are put back in the box until all the n balls are drawn. Find $E(X_i)$ and $V(X_i)$ and show that

$$Cov(X_i, X_j) = \frac{W(W-1)}{(W+B)(W+B-1)} - \frac{W^2}{(W+B)^2}$$

for $1 \leq i \leq j \leq n$

) Write down the variance of $X_1 + X_2 + ... + X_n$ in terms of $Var(X_i)$ and $Cov(X_i, X_j)$ for $1 \leq i \neq j \leq n$. Hence, in part (b), by using $X = X_1 + X_2 + ... + X_n$, find $E(X)$ and $Var(X)$. Is $Var(X)$ in this case smaller than that in (a)?

6. Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{8} x^{\frac{-1}{2}} \exp\left(\frac{-x^{\frac{1}{2}}}{4}\right); & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the

(a) mean,

(b) variance,

(c) median,

(d) inter quartile range.