EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE - 2005/2006

FIRST SEMESTER (Aug./ Sep., 2007)

MT 103 - VECTOR ALGEBRA AND CLASSICAL MECHANICS I

Answer all questions **Time : Three hours**

1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove that the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}.$$

Let $\underline{l}, \underline{m}$, and $\underline{n}$ be three non zero and non co-planner vectors such that any two of them are not parallel. By considering the vector product $(\underline{r} \wedge \underline{l}) \wedge (\underline{m} \wedge \underline{n})$, prove that any vector $\underline{r}$ can be expressed in the form $\underline{r} = (\underline{r} \cdot \underline{\alpha})\underline{l} + (\underline{r} \cdot \underline{\beta})\underline{m} + (\underline{r} \cdot \underline{\gamma})\underline{n}$.

Find the vectors $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ in terms of $\underline{l}, \underline{m}$ and $\underline{n}$.

(b) If a vector $\underline{r}$ is resolved into components parallel and perpendicular to a given vector $\underline{a}$, show that the decomposition is

$$\underline{r} = \frac{(\underline{a} \cdot \underline{r})\underline{a}}{a^2} + \frac{\underline{a} \wedge (\underline{r} \wedge \underline{a})}{a^2}.$$

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2. (a) Define the following terms;

   i. the **gradient** of a scalar field ϕ,

   ii. the **divergence** of a vector field $\underline{F}$,

   iii. the **curl** of a vector field $\underline{F}$.

   (b) Prove that

   i. $\text{div}\,(\phi\,\underline{F}) = \text{grad}\,\phi \cdot \underline{F} + \phi\,\text{div}\,\underline{F}$,

   ii. $\text{curl}\,(\phi\,\underline{F}) = \phi\,\text{curl}\,\underline{F} + \text{grad}\,\phi \wedge \underline{F}$.

   (c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$ and let $\underline{a}$ be a constant vector. Evaluate the following:

   i. $\text{grad}(\underline{a}\,.\,\underline{r})$;

   ii. $\text{curl}(\underline{a} \wedge \underline{r})$.

   Hence show that

   i. $\text{grad}\left(\dfrac{a\,.\,r}{r^3}\right) = \dfrac{a}{r^3} - \dfrac{3(\underline{a}\,.\,\underline{r})}{r^5}\,\underline{r}$,

   ii. $\text{curl}\left(\dfrac{a \wedge r}{r^3}\right) = \dfrac{2a}{r^3} + \dfrac{3\,a \wedge r}{r^5} \wedge \underline{r}$.

3. (a) State the **Stoke's Theorem**.

   Verify the Stoke's theorem for a vector $A = (2x - y)\underline{i} - yz^2\underline{j} - y^2z\underline{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C its boundary.

   (b) State the **Green's Theorem**.

   Verify the Green's theorem in plane for

$$\int_C [(x^2 - xy^3)\,dx + (y^2 - 2xy)dy]$$

   where C is in the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$.

4. Prove that the radial and transverse component of the acceleration of a particle in a plane in terms of polar co-ordinates (r, θ) are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

respectively.

A particle of mass m rests on a smooth horizontal table attached through a fixed point on the table by a light elastic string of modules mg and unstretched length $'a'$. Initially a string is just taut and the particle is projected along the table in a direction perpendicular to the line of the string with velocity $\sqrt{\frac{4ag}{3}}$. Prove that if r is the distance of the particle from the fixed point at time t then

$$\frac{d^2r}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g(r-a)}{a}.$$

Prove that the string will extend until it's length is $2a$ and that the velocity of the particle is then half of it's initial velocity.

5. A particle moves in a plane with velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of the acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

A smooth wire in the form of an arc of a cycloid which equation is $s = 4a\sin\psi$, is fixed in a vertical plane with the vertex downwards and the tangent at vertex horizontal. A small bead of mass m is threaded on the wire and is projected from the vertex with speed $\sqrt{8ag}$. If the resistance of the medium in which the motion take place is $mv^2/8a$ when the speed is v. Show that the bead comes to instantaneous rest at a cusp $(\psi = \pi/2)$ and returns to the starting point with speed $\sqrt{8ga(1-2e^{-1})}$.

6. Establish the equation

$$F(t) = m(t)\frac{dv}{dt} + v_0\frac{dm(t)}{dt}$$

for the motion of a rocket of varying mass $m(t)$ moving in a straight line with velocity $\underline{v}$ under a force $\underline{F}(t)$, matter being emitted at a constant rate with a velocity $\underline{v_0}$ relative to the rocket.

(a) A rocket of total mass m contains fuel of mass ϵm $(0 < \epsilon < 1)$. This fuel burns at a constant rate k and the gas is ejected backward with the velocity u_0 relative to the rocket. Find the speed of the rocket when the fuel has been completely burnt.

(b) A rain drop falls from rest under gravity through a stationary cloud. The mass of the rain drop increases by absorbing small droplets from the cloud. The rate of increment is mrv, where m is the mass, v is the speed and r is a constant. Show that after the rain drop fallen a distance x, $rv^2 = g(1 - e^{-2rx})$.