EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2001/2002)

FIRST SEMESTER

(April/May ' 2002)

MT 302 - COMPLEX ANALYSIS

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Answer all Questions **Time : Three hours**

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1. (a) Define what is meant by "a complex - valued function f, defined on a domain $D(\subseteq \mathbb{C})$, is analytic at a point $z_o \in D$".

   (b) Show that if $z = x + iy$ and a function $f(z) = U(x, y) + iV(x, y)$ is analytic at $z_o = (x_o, y_o)$, then the equations

   $$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \text{ and } \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

   are satisfied at every point of some neighborhood of z_o.

   Give an example to show that the above condition is not sufficient for f to be analytic.

   (c) Find $V(x, y)$ such that $f(z) = U(x, y) + iV(x, y)$ is analytic, where $U(x, y) = e^{-x}(x \sin y - y \cos y)$.

2. (a) Let f be a complex - valued function defined on a domain $D(\subseteq \mathbb{C})$, and let C be a contour.
Define $\int_c f(z)dz$.

(b) Let $M > 0$ be such that $|f(z)| \leq M$ for all z on a contour C and l be the length of C. Show that

$$\left|\int_c f(z)dz\right| \leq Ml.$$

Hence show that

$$\left|\int_c \frac{\log z}{z^2}dz\right| \leq \pi\left(\frac{|\log R| + \pi}{R}\right),$$

where C is the arc of the circle $|z| = R(> 0)$ in the upper half plane from $z = R$ to $z = -R$.

(c) Prove that if a complex - valued function f is analytic and the derivative f' is continuous at all points inside and on a simple closed contour C, then
$\int_c f(z)dz = 0$.
Verify the above result for $f(z) = 3z^2 + iz - 4$ over the contour C given by a square with vertices $1 \pm i$ and $-1 \pm i$.

[Any results used here should be clearly stated.]

3. (a) Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_o is any point interior to C, then prove that

$$f(z_o) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_o} dz.$$

Hence prove that if f is analytic inside and on a circle with centre $z = \alpha$ and radius R, then

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + Re^{i\theta}) d\theta.$$

Evaluate $\frac{1}{2\pi} \int_0^{2\pi} \sin^2(\frac{\pi}{6} + 2e^{i\theta}) d\theta.$

(b) Find the following integrals.

i. $\oint_c \frac{e^z}{z^2(z^2 - 9)} dz$ where C is the circle $C : |z| = 1.$

ii. $\oint_c \frac{z}{(z-1)(z-2)^2} dz$ where C is the circle $C : |z-2| = \frac{1}{2}.$

4. Prove or disprove each of the following statements. Justify your answers.

(a) Every absolutely convergent series of complex numbers is convergent.

(b) $f(z) = \frac{z + i}{\bar{z} - i}$ has a derivative at each point in $\mathbb{C}$.

(c) Every continuous complex-valued function is differentiable.

(d) $f(z) = z^2$ is uniformly continuous in the open disk $|z| < 1$.

(e) If a function f is entire and bounded for all values of z in the complex plane, then the function f is constant throughout the plane.

5. (a) Let f be a complex - valued function and $z_o \in \mathbb{C}$.

Explain what is meant by each of the following statements:

i. f has a pole of order m at z_o;

ii. f has a zero of order m at z_o.

(b) Show that if f is analytic inside and on a simple closed curve C, and f has a pole of order m at $z = \alpha$ then the residue of f at $z = \alpha$ is given by,

$$\lim_{z \to \alpha} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-\alpha)^m f(z)].$$

If $p(z)$ and $q(z)$ are analytic in the neighborhood of z_o, and $q(z)$ has a simple zero at z_o then show that the residue of $\frac{p(z)}{q(z)}$ at z_o is given by $\frac{p(z_o)}{q'(z_o)}$.

(c) Evaluate

$$\int_c \frac{z^2+4}{z^3+2z^2+2z} dz$$

where C is the circle $C : |z+1-i| = \frac{3}{2}$, taken in the positive sense.

6. Prove by contour integration that:

(a) $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}$, if $a > 0$ and $b > 0$;

(b) $\int_0^\infty \frac{\ln(x^2+1)}{(x^2+1)} dx = \pi \ln 2$;

(c) $\int_0^\infty \frac{x \sin 2x}{(x^2+3)} dx = \frac{\pi}{2} e^{-2\sqrt{3}}$.