EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2001/2002)

FIRST SEMESTER

(April/May ' 2002)

MT 303 - FUNCTIONAL ANALYSIS I

Answer all Questions **Time : Two hours**

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1. (a) Define the terms "complete" and "separable" as applied to a normed linear space.

   (b) Show that if $1 \leq p < \infty$, then the sequence space

   $$l^p = \left\{ x = (x_i) : \ x_i \in \mathbb{C}, \ \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

   with norm given by $\| x \| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$ is complete.

   (c) Show that l^p is separable.

   (d) Is the sequence space

   $$l^{\infty} = \left\{ x = (x_i) : \ x_i \in \mathbb{C}, \ \sup_i |x_i| < \infty \right\}$$

   with norm defined by $\| x \| = \sup_i |x_i|$ separable? Justify your answer.

2. (a) Let M be a proper closed subspace of a normed linear space X. Prove that for every real number $r \in (0,1)$, there exists $x_r \in X$ such that $\| x_r \| = 1$ and $\| x_r - m \| > r$ for all $m \in M$.

   (b) Prove that a normed linear space X is of finite dimension if and only if the unit ball $\{x \in X : \| x \| \leq 1\}$ is compact.

3. (a) Define the term " bounded linear operator " from a normed linear space into another normed linear space.

   (b) Let T be a linear operator from a normed linear space X into a normed linear space Y. Show that the following statements are equivalent:

      i. T is continuous at the origin;

      ii. T is continuous on X;

      iii. T is bounded.

   (c) Show that a linear operator $T : X \to Y$ is bounded if and only if T maps bounded subsets of X into bounded subsets of Y.

   (d) Show that the operator $T : l^2 \to l^2$ defined by
   $T(x) = (\eta_i), \ \ \eta_i = \frac{x_i}{2^i}, \ \ x = (x_i)$
   is linear and bounded.

4. State the Hahn-Banach theorem for a normed linear space.

(a) Let M be a proper closed subspace of a normed linear space X.

Let $x_o \in X \setminus M$ and let $d = \inf\{\| x - x_o \| : x \in M\}$.

Show that there exists $f \in X^*$ (dual space of X) such that

$f(x_o) = 1$, $\| f \| = \frac{1}{d}$ and $f = 0$ on M.

Hence show that if $X \neq \{0\}$, then for every $x \in X$, there exists $g \in X^*$ such that $g(x) = \| x \|$ and $\| g \| = 1$.

(b) Let X and Y be normed linear spaces with $X \neq \{0\}$. Show that if $B[X, Y]$ is complete, then Y must be complete.