EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2008/2009)

SECOND SEMESTER (January, 2012)

MT 301 - GROUP THEORY

(Special Repeat)

Answer all questions | Time: Three hours

1. (a) Define the following terms:

   i. group;

   ii. cyclic group;

   iii. abelian group.

   Let G be the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{R}$ with non-zero determinant. Show that $(G, .)$ is a group, where "." denotes usual matrix multiplication.

   Is this an abelian group? Justify your answer.

   (b) i. State and prove the Lagrange's theorem.

   ii. Prove that in a finite group G, the order of each element divides the order of G.

   Hence prove that $x^{|G|} = e, \ \forall \ x \in G$, where e is the identity element of G.

2. (a) What is meant by saying that a subgroup of a group is normal?

   i. Let H and K be two normal subgroups of a group G. Prove that $H \cap K$ is a normal subgroup of G.

ii. Prove that every subgroup of an abelian group G is a normal subgroup of G.

(b) With usual notations prove that:

i. $N(H) \leq G$;

ii. $H \trianglelefteq N(H)$.

(c) Let $Z(G) = \{x \in G \mid xg = gx, \forall\, g \in G\}$. Prove the following:

i. $Z(G) = \bigcap_{a \in G} C(a)$, where $C(a) = \{g \in G \,:\, ga = ag\}$

ii. $Z(G) \trianglelefteq G$.

3. (a) What is meant by an index of a subgroup of a group. Let H and K be two subgroups of a finite group G and $K \subseteq H$. Prove that $[G : K] = [G : H][H : K]$.

(b) State the first isomorphism theorem.

Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove the following:

i. $K \trianglelefteq H$;

ii. $H/K \trianglelefteq G/K$;

iii. $\dfrac{G/K}{H/K} \cong G/H$.

4. (a) Define commutator subgroup G' of a group G.

Prove the following:

i. if $H \trianglelefteq G$ then G/H is abelian if and only if $G' \subseteq H$.

ii. let G be the group of all real 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, where $ad \neq 0$ under matrix multiplication. Show that G' is the set of all matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

(b) What is meant by "*Inn* G (inner automorphism)" of a group G? If H is a subgroup of G, with the usual notations prove that $N(H)/Z(H) \cong Inn\ G$. Hence deduce that $G/Z(G) \cong Inn\ G$.

5. (a) Write down the class equation of a finite group G. Hence or otherwise, prove that the center of G is non-trivial if the order of G is p^n, where p is a prime number and $n \in \mathbb{N}$.

   (b) Define the term $p-$ group.

   Let G be a finite abelian group and p be a prime number which divides the order of G. Prove that G has an element of order p.

6. (a) Define the following terms as applied to a group:

   i. permutation;

   ii. cycle of order r;

   iii. transposition.

   (b) Prove that the permutation group on n symbols (S_n) is a finite group of order $n!$.

   Is S_n an abelian group for $n > 2$? Justify your answer.

   (c) Prove that the set of even permutations forms a normal subgroup of S_n. Hence prove that S_n/A_n is a cyclic group of order 2.