# EASTERN UNIVERSITY, SRI LANKA

## THIRD EXAMINATION IN SCIENCE 2003/2004

## FIRST SEMESTER

## Oct/Nov 2004

## MT 306 - PROBABILITY THEORY

**Answer all questions**

**Time : Two hours**

1. Let $Y_1$ and $Y_2$ have a joint density function given by

$$f(y_1, y_2) = \begin{cases} 3y_1 & ; 0 \leq y_2 \leq y_1 \leq 1, \\ 0 & ; \text{elsewhere.} \end{cases}$$

   (a) Find the marginal density functions of $Y_1$ and $Y_2$.

   (b) Find the conditional density function of $Y_1$ given $Y_2 = y_2$.

   (c) Find $P(Y_1 \leq 3/4 | Y_2 \leq 1/2)$.

   (d) Let $X = Y_1 - Y_2$, Find $E(X)$ and $V(X)$.

2. (a) Define the probability function.

   i. Show that probability of exactly one of the events $A$ or $B$ occurs is $P(A) + P(B) - 2P(A \cap B)$.

   ii. Prove that

$$P(\bigcap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

   where $A_i (i = 1, 2, \cdots, n)$ are events defined on the sample space.

(b) Define the Moment generating function.

i. Show that if $X$ and $Y$ are independent random variables with moment generating functions $M_X(t)$ and $M_Y(t)$ respectively, then the moment generating function of $X+Y$ is $M_{X+Y}(t) = M_X(t)M_Y(t)$.

ii. The probability density function of a Gamma distribution with parameters $m$ and $\lambda$ is given by

$$f_X(x) = \begin{cases} \dfrac{\lambda^m x^{m-1} e^{-\lambda x}}{\Gamma(m)} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the moment generating function of $X$. Suppose that the random variable $Y$ which is independent of $X$ has a Gamma distribution with parameters $s$ and $\lambda$. Show that $X+Y$ has a Gamma distribution with parameters $s+m$ and $\lambda$.

3. (a) If $X$ is a random variable having a binomial distribution with the parameters $n$ and $\theta$, then show that the moment generating function of $Z = \dfrac{X - n\theta}{\sqrt{n\theta(1-\theta)}}$ approaches that of the standard normal distribution when $n \longrightarrow \infty$.

(b) Suppose that the random variable X is uniformly distributed on (0,1). Assume that the conditional distributional $Y/X = x$ has a binomial distribution with parameters $n$ and $p = x$.

That is, $P(Y = y/X = x) = \begin{pmatrix} n \\ y \end{pmatrix} x^y(1-x)^{n-y}$ ; $y = 0, 1, \cdots, n.$

Find

i. E(Y).

ii. Find the distribution of Y.

4. (a) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a distribution with probability density function,

$$f(x) = \begin{cases} \frac{1}{\theta} exp(\frac{-x}{\theta}) \; ; & 0 < x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

where $\theta > 0$ is an unknown parameter.

i. Find $\hat{\theta}$, the method of moments estimator of $\theta$.

ii. Verify that $\hat{\theta}$ is an unbiased estimator of $\theta$ and find its variance. State with reasons whether $\hat{\theta}$ is consistent for $\theta$.

iii. Find the Cramer-Rao lower bound for the variance of unbiased estimators of $\theta$ and deduce the efficiency.

(b) The shopping times were recorded for n=64 randomly selected customers for a local supermarket. The average and variance of the 64 shopping times were 33 minutes and 256, respectively. Find the 90% confidence interval for the true average shopping time per customer.

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