EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE, (2002/2003)

(June/July, 2003)

REPEAT

FIRST SEMESTER

MT 302 - COMPLEX ANALYSIS

Answer all questions Time allowed: **3 Hours**

Q1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$. **[20]**

(b) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$ be differentiable at some $z_0 = x_0 + i\,y_0 \in A$. If $f(z) = u(x, y) + i\,v(x, y)$, then prove that

(i) $u(x, y)$ and $v(x, y)$ have partial derivatives at $z_0 = x_0 + i\,y_0$ that satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

[50]

(c) Obtain a harmonic conjugate $v(x, y)$ of a harmonic function $u(x, y) = y^3 - 3x^2y$ such that $f(z) = u(x, y) + i\,v(x, y)$ is analytic. **[30]**

Q2. (a) For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)\,dz$. **[20]**

(b) Let γ be a path and let f be a continuous function on γ. If $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \gamma$, then prove that

$$\left|\int_\gamma f(z)\,dz\right| \leq ML,$$

where $L := length(\gamma)$. [50]

Hence show that

$$\left|\int_{\gamma} \frac{z^{\frac{1}{2}}}{z^2+1}\, dz\right| \leq \frac{3\sqrt{3}}{8}\pi,$$

where γ is the semi circular path given by $z = 3e^{i\theta}$, $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

[30]

Q3. (a) State the **Cauchy's Integral Formula.** [30]

By using the **Cauchy's Integral Formula** compute the following integrals:

(i) $\displaystyle\int_{C(0;1)} \frac{1}{(z-\frac{1}{2})(z+2)^2}\, dz;$ [20]

(ii) $\displaystyle\int_{C(0;2)} \frac{\cos z}{(z^3+9z)}\, dz,$ [20]

where $C(a;r)$ denotes a positively oriented circle centre a with radius r.

(b) Prove the **Mean Value Property for Analytic Functions**: let f be analytic on a disc $D(a;r) := \{z \in \mathbb{C} : |z-a| < r\}$ and let $s \in (0,r)$. Then

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+se^{it})\, dt.$$

[30]

Q4. (i) Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being **entire.** [20]

(ii) Prove **Liouville's Theorem**: if f is entire and

$$\frac{\max\{|f(t)| : |t| = r\}}{r} \to 0, \quad \text{as} \quad r \to \infty,$$

then f is constant. [50]

(You may assume any results without proof).

(c) State the **Maximum-Modulus Theorm.** [30]

Q5. If f has a pole of order m at z_0, then prove that the residue of f at z_0 denoted by $Res(f; z_0)$ and is given by

$$Res(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)].$$

[60]

Hence evaluate the following integral

$$\int_C \frac{dz}{z^3(z+4)}$$

taken counterclockwise around the circle

(i) $|z| = 2$; [20]

(ii) $|z+2| = 3$. [20]

Q6. (a) State the **Principle of the Argument** theorem. [20]

(b) Prove **Rouche's Theorem:** let γ be a simple closed path in an open starset A. Suppose that

(i) f, g are analytic in A except for finitely many poles, none lying on γ.

(ii) f and $f+g$ have finitely many zeros in A.

(iii) $|g(z)| < |f(z)|, \quad z \in \gamma$. Then

$$ZP(f+g; \gamma) = ZP(f; \gamma)$$

where $ZP(f+g; \gamma)$ and $ZP(f; \gamma)$ denote the number of zeros – number of poles inside γ of $f+g$ and f respectively, where each is counted as many times as its order. [40]

(c) State the **Fundamental theorem of Algebra.** [20]

(d) Prove that all 5 zeros of $P(z) = z^5 + 3z^3 + 1$ lie in $|z| < 2$. [20]