EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE (2004/2005)

FIRST SEMESTER (Jan./ Feb., 2006)

MT 201 - VECTOR SPACES AND MATRICES

Answer all questions | Time allowed : Three hours

1. (a) What is meant by a vector space?

   (b) Let V be a vector space over the field F and W be a non-empty subset of V. Prove that W is a subspace of V if and only if $ax + by \in W$ for every $x, y \in W$ and for every $a, b \in F$.

   (c) Let $V \times W$ be the Cartesian product of two vector spaces V and W. Define addition and scalar multiplication on $V \times W$ as follows:

   $$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \quad (v_i, w_i) \in V \times W, \quad i = 1, 2;$$

   $$\alpha(v, w) = (\alpha v, \alpha w), \quad \alpha \in \mathbb{R}, (v, w) \in V \times W.$$

   Prove that, $V \times W$ is a vector space.

   (d) Let V be the vector space of all 2×2 matrices over the field $\mathbb{R}$. Verify whether the following set W is a subspace of V, where,

   i. W consists of all matrices with zero determinant,

   ii. W consists of all matrices A, for which $A^2 = A$.

2. (a) Define the following in a vector space V.

   i. Span of S, where $S \subseteq V$,

   ii. Direct sum of two subspaces S_1 and S_2 in V.

   (b) Let V be vector space over the field F.

   i. Let u_0 and v_0 be linearly independent vectors in V and let $u_1 = au_0 + bv_0$ and $v_1 = cu_0 + dv_0$, where $a, b, c, d \in F$. Show that u_1 and v_1 are linearly independent if and only if $ad - bc \neq 0$.

   ii. Show that, $S \cup \{v\}$ is linearly independent if and only if $v \notin \langle S \rangle$, where S is a linearly independent subset of V and $v \in V$.

   (c) Let v_1 and v_2 belong to the vector space V over the field F. Show that;

   i. $\langle \{v_1 + v_2, v_2\} \rangle = \langle \{v_1, v_2\} \rangle$,

   ii. $\langle \{v_1 + v_2, v_1 - v_2\} \rangle = \langle \{v_1, v_2\} \rangle$, if $F = \mathbb{Q}$, a set of rational numbers.

   (d) Let V be the vector space of $n-$square matrices over the field $\mathbb{R}$. Let U and W be the subspaces of symmetric and antisymmetric matrices, respectively. Show that $V = U \oplus W$.

3. (a) Suppose $\{v_1, v_2, \cdots, v_n\}$ generates a vector space V. If $\{w_1, w_2, \cdots, w_m\}$ is linearly independent then prove that $m \leq n$ and V is generated by a set of the form $\{w_1, w_2, \cdots, w_m, v_{i_1}, v_{i_2}, \cdots, v_{i_{n-m}}\}$.

   Use this result to prove, a linearly independent set which contains number of elements equal to the dimension of the vector space is a basis for that vector space.

   (b) State the dimension theorem for two subspaces of a finite dimensional vector space V.

   i. Let U_1 and U_2 be be subspaces of the vector space V. If $\dim U_1 = 3$, $\dim U_2 = 4$, $\dim V = 6$, show that $U_1 \cap U_2$ contains a non-zero vector. If $\dim U_1 = 2$, $\dim U_2 = 4$, $\dim V, = 6$ show that $U_1 + U_2 = V$ if and only if $U_1 \cap U_2 = \{0\}$.

ii. Let U_1 and U_2 be distinct $(n-1)-$dimensional subspaces of a $n-$dimensional vector space V. Show that $\dim(U_1 \cap U_2) = n-2$. If $U_1, U_2, \cdots, U_t$ are $(n-1)-$dimensional subspaces of V, prove that $\dim(U_1 \cap U_2 \cdots \cap U_t) \geq n-t$.

4. Define the term "non-singular matrix".

(a) The $m \times m$, $n \times n$, $m \times n$ matrices P, Q, A over the field F are placed together as shown to form the $(m+n) \times (m+n)$ matrix

$X = \begin{pmatrix} P & A \\ O & Q \end{pmatrix}$, where O denotes the $n \times m$ zero matrix. Show that X is non-singular if and only if P and Q are non-singular, in which case

$$X^{-1} = \begin{pmatrix} P^{-1} & -P^{-1}AQ^{-1} \\ O & Q^{-1} \end{pmatrix}.$$

(b) Show that a matrix over F of the form

$\begin{pmatrix} P & O & O \\ A & Q & O \\ B & C & R \end{pmatrix}$, where $P, \bar{Q}, R$ are non-singular,

is itself non-singular and express its inverse in a similar (partitioned) form.

(c) i. Show that the 3×3 matrix A over the field F commutes with

$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ if and only if $A = aI + bB + cB^2$, where $a, b, c \in F$ and I is the identity matrix of order 3.

ii. Let $A = (a_{ij})$ be a diagonal $n \times n$ matrix over F with n distinct diagonal entries a_{ii}. Show that the $n \times n$ matrix B over F commutes with A if and only if B is diagonal.

5. (a) Let T be a linear transformation from a vector space V into another vector space W. Define

i. range space $R(T)$ and

ii. null space $N(T)$.

(b) Let V be a vector space with finite dimension, and let $T : V \to U$ be a linear transformation with range U' and kernel W. Prove that $\dim U' + \dim W = \dim V$.

(c) Let V and V' be vector spaces over the field F. Let V has basis $\{v_1, v_2, \cdots, v_m\}$ and let $\{v'_1, v'_2, \cdots, v'_m\}$ be any vectors in V'. Then prove that there is a unique linear transformation $T : V \to V'$ such that

$T(v_1) = v'_1, T(v_2) = v'_2, \cdots, T(v_m) = v'_m$. Further prove that T is injective if and only if $\{v'_1, v'_2, \cdots, v'_m\}$ is linearly independent.

6. (a) Let $A = (a_{ij})$ be an $n \times n$ matrix. Define adjoint of A.

Prove that $A.\, \text{adj}A = \text{adj}A.\, A = (\text{det}A).I$, where I is the $n \times n$ identity matrix. If A is a non-singular matrix, then show that

i. $\text{adj}(\text{adj } A) = (\text{det}A)^{n-2}$

ii. $\text{adj}(\text{adj}(\text{adj } A)) = (\text{det}A)^{n^2-3n+3}$

(b) Consider a system of linear equations with the same number of equations as unknowns.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots &= \cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

i. Suppose that associated homogeneous system has only the zero solution. Show that this system has a unique solution for every choice of constants b_i.

ii. Suppose that associated homogeneous system has a non-zero solution. Show that there are constants b_i for which this system does not have a solution. Also show that if system has a solution, then it has more than one.