EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 2004/2005

FIRST SEMESTER (Jan./ Feb., 2006)

MT 203 - EIGEN SPACE AND QUADRATIC FORMS

Answer all questions **Time allowed: Two hours**

1. Define

- Geometric multiplicity,
- Algebraic multiplicity;

of an eigenvalues λ of a linear transformation on a finite dimensional vector space V.

(a) Let A be an $n \times n$ non-singular matrix and let $\chi_A(t)$ denote the characteristic polynomial of A.

Show that

$$\chi_{A^{-1}}(t) = \frac{(-t)^n}{detA}\ \chi_A(1/t); \qquad (t \neq 0).$$

Deduce that if $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A with algebraic multiplicities then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1} with algebraic multiplicities.

(b) Let A be a matrix of order n such that $A^2 = I$. Show that every eigenvalue of A is 1 or -1.

Let $A=\begin{pmatrix} 1 & 1 & 1 \\ -4 & -4 & -1 \\ 4 & 1 & -2 \end{pmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is diagonal.

2. Define the term " minimum polynomial " of a square matrix.

(a) Prove that the characteristic polynomial of an $n \times n$ matrix A always divides the n^{th} power of its minimum polynomial.

(b) Let $B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$ be a block diagonal matrix, where B_{11} and B_{22} are square matrices. Show that the minimum polynomial $m(t)$ of B is the least common multiple of the minimum polynomials $g(t)$ and $h(t)$ of B_{11} and B_{22} respectively.

(c) State the **Cayley-Hamilton theorem**

Find the minimum polynomial of the matrix A given by

$$A=\begin{pmatrix} 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

3. (a) Find an orthogonal transformation which reduces the following quadratic form to a diagonal form

$$3x_1^2 + 2x_2^2 + 4x_3^2 - 4x_1x_2 + 4x_1x3.$$

(b) Simultaneously diagonalize the following pair of quadratic forms;

$$x_1^2 - x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_1x_3$$

$$3x_1^2 + x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_2x_3 - 2x_1x_3.$$

4. What is meant by an " inner product " on a vector space?

(a) Verify that the function $< . , . >$, define by

$$< x, y >= \sum_{i=1}^{n} x_i \, \overline{y_i} \; , \qquad x, y \in \mathbb{C}^n$$

is an inner product on $\mathbb{C}^n$.

(b) State **Gram-Schmidt Process** and use it to find the othonormal set for span of S in $\mathbb{R}^4$, where $S = \{ (1,0,-1,0)^\top , (0,1,2,1)^\top , (2,1,-1,0)^\top \}$.

(c) Let V be an inner product space and W be a subspace of V.

i. Show that there is an orthonormal basis of Wwhich is part of an orthonormal basis of V.

ii. Prove that, $V = W \oplus W^\perp$, where $W^\perp$ is orthogonal complement of W and $\oplus$ denotes the direct sum.