EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE (2004/2005)

FIRST SEMESTER (Jan./ Feb., 2006)

MT 207 - NUMERICAL ANALYSIS

Answer all questions Time allowed : Two hours

1. Define "**absolute error**" and "**relative error**" of a numerical value.

   (a) i. Let x_1 and x_2 be approximations to the true values X_1 and X_2 respectively and e_1 and e_2 be the corresponding errors in these approximation. If e_1 is due to rounding - off X_1 to k_1 decimal places and e_2 is due to rounding - off X_2 to k_2 decimal places, find a bound on the absolute error in x_1x_2.

   ii. The numbers a and b when rounded to four significant digits are 37.26 and 0.02146 respectively. Evaluate an approximation to ab and discuss the error involved.

   (b) i. For a quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$ and $b^2 - 4ac > 0$, the roots can be computed with the formulas

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

   Show that these roots can be calculated with the equivalent formulas

$$x_1 = \frac{-2c}{b + \sqrt{b^2 - 4ac}} \quad \text{and} \quad x_2 = \frac{-2c}{b - \sqrt{b^2 - 4ac}}.$$

ii. Use the appropriate formula for computing x_1 and x_2 mentioned in part(i) to find the roots of the quadratic equation

$$x^2 - 60x + 1 = 0,$$

using 4 significant digits throughout the calculation.

2. (a) Define the order and the asymptotic error constant of the iterative method to compute the nonlinear equation $f(x) = 0$.

   i. Obtain Newton-Raphson method to compute the roots of the equation $f(x) = 0$ in an interval $[a, b]$.
   Solve $\sin x = 1 + x^3$ using Newton-Raphson method.

   ii. Show that the asymptotic error constant of the Newton-Raphson method is $\dfrac{1}{2}\dfrac{f''(\alpha)}{f'(\alpha)}$.

   (b) Suppose that $g(x)$ and $g'(x)$ are defined and continuous on (a, b) and p_0, p_1, p_2 lie in this interval and $p_1 = g(p_0)$, $p_2 = g(p_1)$. Also, assume that there exists a constant K so that $|g'(x)| < K$. Show that $|p_2 - p_1| < K\,|p_1 - p_0|$.

   (c) Let $g(x) = 1.5 + 0.5x$ and $p_0 = 4$ and consider fixed-point iteration.

   i. Show that the fixed point is $p = 3$.

   ii. Show that $|p - p_n| = |p - p_0|\,/2^n$ for $n = 1, 2, ...$

3. (a) Let $f(x)$ be an $(n+1)$ times continuously differentiable function of x and $y_0, y_1, \ldots, y_n$ be the values of $f(x)$ at $x = x_0, x_1, \ldots, x_n$ respectively.

i. Derive the Lagrange's Interpolation polynomial $P_n(x)$ to estimate the value of $f(x)$ for any $x \in [x_0, x_n]$, in the form

$$P_n(x) = \sum_{i=0}^{n} \frac{\Pi(x) y_i}{(x - x_i)\Pi'(x_i)},$$

where $\Pi(x) = \prod_{i=0}^{n} (x - x_i)$.

ii. Using Lagrange's Interpolation formula, express the function

$$\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$$

as a sum of partial functions.

iii. Use the special function

$$g(t) = f(t) - P_n(t) - E_n(x)\frac{(t - x_0)(t - x_1) \ldots (x - x_n)}{(x - x_0)(x - x_1) \ldots (x - x_n)},$$

to prove that the error term $E_n(x) = f(x) - P_n(x)$ in the interpolating polynomial, has the form

$$E_n(x) = (x - x_0)(x - x_1) \ldots (x - x_n)\frac{f^{(n+1)}(c)}{(n+1)!}.$$

Hence show that

$$|E_1(x)| \leq \frac{h^2 M_2}{8}$$

for the equally spaced nodes $x_k = x_0 + kh$, where $|f^{(n+1)}(x)| \leq M_{n+1}$ for $x_0 \leq x \leq x_n$.

4. (a) Let $I(f) = \int_a^b f(x)dx$.

$I(P_n)$ is the approximation to $I(f)$ where $P_n(x)$ is the polynomial of degree$\leq n$ which interpolates $f(x)$ at the equispaced nodes $a \equiv x_0, x_1, \ldots, x_n \equiv b$, where $x_k = x_0 + kh$ for $k = 0, 1, \ldots, n$ and $x_i \in [a, b]$ for $i = 0, 1, \ldots, n$. The error in the approximation is given by,

$$E(f) = I(f) - I(P_n).$$

Obtain the composite Trapezoidal rule and show that the composite error is

$$-\frac{(b-a)}{12}h^2 y''(\eta), \qquad \eta \in [a, b].$$

Evaluate the integral

$$\int_0^1 \frac{dx}{1+x^2}$$

using Trapezoidal rule by taking $h = 1/4$. Hence compute the approximate value of π.

(b) Describe Gauss Elimination with scaled partial pivoting .

Suppose that

$$\begin{aligned} 2x_1 &+ x_2 &+ 3x_3 &= 1 \\ 4x_1 &+ 6x_2 &+ 8x_3 &= 5 \\ 6x_1 &+ \alpha x_2 &+ 10x_3 &= 1 \end{aligned}$$

with $|\alpha| < 10$. For which of the following values of α will there be no row interchange required when solving this system using scaled partial pivoting?

i. $\alpha = 6$

ii. $\alpha = 9$

iii. $\alpha = -3$