EASTERN UNIVERSITY, SRI LANKA

THIRD YEAR EXAMINATION IN SCIENCE 2002/2003

SECOND SEMESTER

(April/May '2004)

(Re-Repeat)

MT 302 - COMPLEX ANALYSIS

Answer five questions only

Time : Three hours

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1. (a) Define what is meant by " a complex-valued function f has a limit at $z_0 \in \mathbb{C}$".

   (b) Show that
$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad \text{for } |z| < 1.$$

   Deduce that if $0 < r < 1$, then
$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2},$$

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta - r^2}{1 - 2r\cos\theta + r^2}.$$

2. (a) What is meant by saying that a complex-valued function f, defined on a domain D $(\subseteq \mathbb{C})$, is analytic at a point $z_0 \in D$.

   Show that if $z = x + iy$ and a function $f(z) = u(x, y) + iv(x, y)$ is analytic at $z_0 = x_0 + iy_0$, then the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied at every point of some neighbourhood of z_0.

(b) Prove that the function $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic.

Find a function $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic.

3. Let $M > 0$ be such that $|f(z)| \leq M$ for all z on a contour C and l be the length of C.

Show that

$$\left| \int_C f(z)\, dz \right| \leq Ml \,.$$

Hence show that

$$\left| \int_C \frac{z^{1/2}}{z^2+1}\, dz \right| \leq \frac{3\sqrt{3}}{8}\pi ,$$

where C is the semi circular path given by $z = 3e^{i\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

4. Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then prove that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \,.$$

Prove that if $f(z)$ is analytic inside and on the circle C if radius r with centre at $z = z_0$, then

$$\left| f^{(n)}(z_0) \right| \leq \frac{Mn!}{r^n} , \quad n = 0, 1, 2, \cdots$$

where M is a positive constant such that $|f(z)| \leq M$ for all z inside and on C.

5. Prove or disprove each of the following statements. Justify your answer.

   (a) If $f(z)$ and $\overline{f(z)}$ are analytic functions in a domain D, then $f(z)$ is a constant in D.

   (b) The function $f(z) = \dfrac{1}{z}$ is uniformly continuous in $|z| < 1$.

   (c) Every polynomial of degree n with complex coefficients, has exactly n zero.

   (d) The function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = |z|^2$, has derivative at each point in $\mathbb{C}$.

6. (a) Let f be a complex-valued function and $z_0 \in \mathbb{C}$. Explain what is meant by each of the following statements:

   (i) f has a pole of order m at z_0 ;

   (ii) residue of f at z_0.

   (b) Show that if f is analytic inside and on a simple closed contour C and f has a pole of order m at $z = \alpha$ then the residue of f at $z = \alpha$ is given by

$$\lim_{z \to \alpha} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-\alpha)^m \, f(z) \right\}.$$

7. (a) State and prove the Argument Theorem.

   (b) (i) If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then show that both functions $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

   (ii) Show that all the roots of $2z^5 - z^3 + z + 7 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

8. (a) State the Residue Theorem.

(b) Find the value of the integral

$$\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$$

where C is taken counter clockwise around the circle

(i) $|z-2|=2$;

(ii) $|z|=4$.