EASTERN UNIVERSITY, SRI LANKA Wersity, Sri b

THIRD EXAMINATION IN SCIENCE, (2005/2006)

(August/September, 2007)

FIRST SEMESTER

PROPER & REPEAT MT 302 - COMPLEX ANALYSIS'

Answer all questions Time allowed: 3 Hours

- (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f: A \to \mathbb{C}$. Define what is meant Q1. 20 by f being analytic at $z_0 \in A$.
 - (b) Let the function f(z) = u(x,y) + iv(x,y) be defined throughout some ϵ neighbourhood of a point $z_0 = x_0 + i y_0$. Suppose that the first-order partial derivatives of u(x,y) and v(x,y) with respect to x and y exist everywhere in that neighbourhood and that they are continuous at (x_0, y_0) . Prove that if those partial derivatives satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at (x_0, y_0) , then the derivative $f'(z_0)$ exists.

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(c) Determine where f'(z) exists and find its value for

$$f(z) = z \operatorname{Im} z.$$

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Q2. (a) (i) Define what is meant by a path $\gamma : [\alpha, \beta] \to \mathbb{C}$. [10]

(ii) For a path γ and a continuous function $f: \gamma \to \mathbb{C}$, define $\int_{\gamma} f(z) dz$. [10]

(b) Let $a \in \mathbb{C}$, r > 0, and $n \in \mathbb{Z}$. Show that

$$\int_{C(a; r)} (z - a)^n dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1 \end{cases}$$

where C(a; r) denotes a positively oriented circle with centre a and radius r. [30]

(State any results you use without proof).

(c) State the Cauchy's Integral Formula. [20]

By using the Cauchy's Integral Formula compute the following integrals:

(i)
$$\int_{C(0,2)} \frac{\sin z}{z^2 + 1} dz;$$
 [15]

(ii)
$$\int_{C(0,3)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz.$$
 [15]

- Q3. (a) State the Mean Value Property for Analytic Functions. [10]
 - (b) (i) Define what is meant by the function $f: \mathbb{C} \to \mathbb{C}$ being entire.

 [10]
 - (ii) Prove Liouville's Theorem: If f is entire and

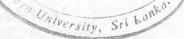
$$\frac{\max\left\{|f(t)|:|t|=r\right\}}{r}\to 0,\quad \text{as } r\to \infty,$$

then f is constant.

[30]

(State any results you use without proof)

Let f(z) = u(x, y) + iv(x, y) be an entire function and that u(x, y) has an upper bound for all (x, y) in the xy plane. Show that u(x, y) is constant throughout the plane. [10]



(c) Prove the Maximum-Modulus Theorm: Let f be analytic in an open connected set A. Let γ be a simple closed path that is contained, together with its inside, in A. Let

$$M := \sup_{z \in \gamma} |f(z)|.$$

If there exists z_0 inside γ such that $|f(z_0)| = M$, then f is constant throughout A. Consequently, if f is not constant in A, then

$$|f(z)| < M \ \forall \ z_0 \text{ inside } \gamma.$$

[40]

(State any theorem you use without proof)

- Q4. (a) Let $\delta > 0$ and let $f: D^*(z_0; \delta) \to \mathbb{C}$, where $D^*(z_0; \delta) := \{z: 0 < |z z_0| < \delta\}$. Define what is meant by
 - (i) f having a singularity at z_0 ;
 - (ii) the order of f at z_0 ;
 - (iii) f having a pole or zero at z_0 of order m;
 - (iv) f having a simple pole or simple zero at z_0 .

[40]

(b) Prove that

$$ord(f; z_0) = m$$

if and only if

$$f(z) = (z - z_0)^m g(z), \quad \forall z \in D^*(z_0; \delta),$$

for some $\delta > 0$, where g is analytic in $D(z_0; \delta)$ and $g(z_0) \neq 0$.

[60]

Q5. (a) Prove that if f has a simple pole at z_0 , then

$$Res(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z_0).$$

[30]

(b) Let f be analytic in $\{z: Im(z) \geq 0\}$, except possibly for finitely many singularities, none on the real axis. Suppose there exist M, R > 0 and $\alpha > 1$ such that

$$|f(z)| \le \frac{M}{|z|^{\alpha}}, \quad |z| \ge R \text{ with } Im(z) \ge 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x) \, dx$$

converges (exists) and

 $I = 2\pi i \times \text{Sum}$ of Residues of f in the upper half plane.

[50]

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} \, dx.$$

[20]

(You may assume without proof the Residue Theoerem).

- Q6. (a) State the Argument Theorem. [20]
 - (b) Prove Rouche's Theorem: Let γ be a simple closed path in an open starset A. Suppose that
 - (i) f, g are analytic in A except for finitely many poles, none lying on γ .
 - (ii) f and f + g have finitely many zeros in A.
 - (iii) $|g(z)| < |f(z)|, z \in \gamma$. Then

$$ZP(f+g;\gamma) = ZP(f;\gamma)$$

where $ZP(f+g;\gamma)$ and $ZP(f;\gamma)$ denote the number of zeros – number of poles inside γ of f+g and f respectively, where each is counted as many times as its order. [40]

- (c) State the Fundamental theorem of Algebra. [20]
- (d) Prove that the equation $2e^z + z + 3 = 0$ has exactly one root in the left-half plane. [20]