



EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2005/2006

FIRST SEMESTER (Aug./Sep., 2007)

MT 306 - PROBABILITY THEORY

(Proper & Repeat)

Answer all questions

Time : Two hours

- Q1. (a) i. State and prove the Baye's theorem.  
ii. In a certain college, 4% of the men and 1% of the women are taller than 1.8m. Furthermore 60% of the students are women. If a student selected at random is taller than 1.8m, what is the probability that the student is a woman?

- (b) A random variable  $X$  has Poisson distribution with parameter  $\lambda$  given by

$$P[X = x] = \frac{e^{-\lambda} \lambda^x}{x!}.$$

Find the mean, variance and the moment generating function of  $X$ .

- (c) The mean number of bacteria per milliliter of a liquid is known to be 4. Assuming that the number of bacteria follows a Poisson distribution, find the probability that
- in 1 ml of liquid there will be no bacteria,
  - in 3 ml of liquid there will be less than two bacteria,
  - in  $\frac{1}{2}$  ml of liquid there will be more than two bacteria.

- Q2. (a) If  $X$  is a random variable with density function  $f_X$  and  $g(x)$  is a monotonically increasing and differentiable function from  $\mathbb{R}$  to  $\mathbb{R}$ , show that  $Y = g(X)$  has the density function

$$f_Y(y) = f_X[g^{-1}(y)] \frac{d}{dy}[g^{-1}(y)], \quad y \in \mathbb{R}.$$

- (b) Let  $X$  be a random variable with exponential distribution with parameter  $\lambda$ . Find the density function of

- i.  $2X + 5$ ,
- ii.  $(1 + X)^{-1}$ .

- (c) Random variable  $X$  and  $Y$  have joint density function

$$f_{XY}(x, y) = \begin{cases} k(x^3 + 1)y & \text{if } 0 < x < 1, \quad 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find

- i. the value of  $k$ ,
- ii. marginal density functions of  $X$  and  $Y$ ,
- iii.  $E(XY)$ ,
- iv. Are  $X$  and  $Y$  independent?

- Q3. (a) Define the Moment Generating Function of a random variable  $X$ .

Find the moment generating function of the Gamma distribution given by

$$f(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} & ; \quad x \geq 0 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

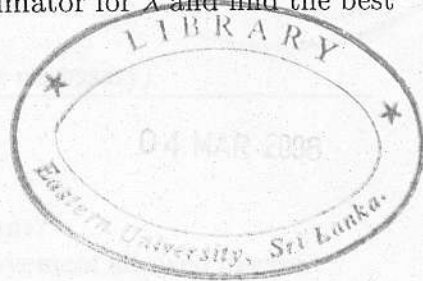
Hence find the mean and variance.

- (b) i. Define the following terms:

- \* Unbiased estimator,
- \* Risk function.

ii. Let  $X_1, X_2, \dots, X_n$  be a random sample from a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Determine  $c$  such that  $c[(X_1 - X_2)^2 + (X_3 - X_4)^2 + (X_5 - X_6)^2]$  is an unbiased estimator for  $\sigma^2$ .

iii. Let  $X_1, X_2, \dots, X_n$  be a random sample from Poisson distribution with parameter  $\lambda$ . Let  $T_1 = \frac{X_i + X_j}{2}$  and  $T_2 = \frac{1}{n} \sum_{i=1}^n X_i$  where  $1 \leq i \leq n$ ,  $1 \leq j \leq n$ . Show that  $T_1$  and  $T_2$  are unbiased estimator for  $\lambda$  and find the best estimator for  $\lambda$ .



Q4. (a) Define the maximum likelihood estimator.

Determine the maximum likelihood estimators of the parameters of the following distributions:

- i. Exponential distribution with parameter  $\theta$ ,
- ii. Normal distribution with mean  $\mu$  and variance  $\sigma^2$ .

(b) Let  $X_1, X_2, \dots, X_n$  be  $n$  a random sample from a normal distribution with unknown mean  $\mu$  and known variance  $\sigma^2$ . Find  $100(1 - \alpha)\%$  confidence interval for  $\mu$ .

(c) On the basis of results obtained from a random sample of 100 men from a particular district, the 95% confidence interval for the mean height of the men in the district is found to be (177.22 cm, 179.18 cm). Find the value of  $\bar{X}$ , the mean of the sample, and  $\sigma^2$ , the standard deviation of the normal population from which the sample is drawn. Calculate the 98% confidence interval for the mean height.