# EASTERN UNIVERSITY, SRI LANKA

## THIRD EXAMINATION IN SCIENCE - 2005/2006

## FIRST SEMESTER (Aug./Sep.,2007)

## MT 306 - PROBABILITY THEORY

(Proper & Repeat)

Answer all questions Time : Two hours

Q1. (a) i. State and prove the Baye's theorem.

ii. In a certain college, 4% of the men and 1% of the women are taller than 1.8 m. Furthermore 60% of the students are women. If a student selected at random is taller than 1.8 m, what is the probability that the student is a woman?

(b) A random variable $X$ has Poisson distribution with parameter $\lambda$ given by

$$P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!}.$$

Find the mean, variance and the moment generating function of $X$.

(c) The mean number of bacteria per milliliter of a liquid is known to be 4. Assuming that the number of bacteria follows a Poisson distribution, find the probability that

i. in 1 ml of liquid there will be no bacteria,

ii. in 3 ml of liquid there will be less than two bacteria,

iii. in $\frac{1}{2}$ ml of liquid there will be more than two bacteria.

Q2. (a) If $X$ is a random variable with density function $f_X$ and $g(x)$ is a monotonically increasing and differentiable function from $\mathbb{R}$ to $\mathbb{R}$, show that $Y = g(X)$ has the density function

$$f_Y(y) = f_X[g^{-1}(y)]\,\frac{d}{dy}[g^{-1}(y)], \quad y \in \mathbb{R}.$$

(b) Let $X$ be a random variable with exponential distribution with parameter $\lambda$. Find the density function of

i. $2X + 5$,

ii. $(1 + X)^{-1}$.

(c) Random variable $X$ and $Y$ have joint density function

$$f_{XY}(x, y) = \begin{cases} k(x^3 + 1)y & \text{if } 0 < x < 1, \quad 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find

i. the value of $k$,

ii. marginal density functions of $X$ and $Y$,

iii. $E(XY)$,

iv. Are $X$ and $Y$ independent?

Q3. (a) Define the Moment Generating Function of a random variable $X$.

Find the moment generating function of the Gamma distribution given by

$$f(x) = \begin{cases} \dfrac{\lambda^n\, x^{n-1}\, e^{-\lambda x}}{\Gamma(n)} & ; \quad x \geqslant 0 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

Hence find the mean and variance.

(b) i. Define the following terms:

* Unbiased estimator,

* Risk function.

ii. Let $X_1, X_2, ..., X_n$ be a random sample from a normal distribution with mean $\mu$ and variance $\sigma^2$. Determine $c$ such that $c[(X_1 - X_2)^2 + (X_3 - X_4)^2 + (X_5 - X_6)^2]$ is an unbiased estimator for $\sigma^2$.

iii. Let $X_1, X_2, ..., X_n$ be a random sample from Poisson distribution with parameter $\lambda$. Let $T_1 = \dfrac{X_i + X_j}{2}$ and $T_2 = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. Show that $T_1$ and $T_2$ are unbiased estimator for $\lambda$ and find the best estimator for $\lambda$.



Q4. (a) Define the maximum likelihood estimator.

Determine the maximum likelihood estimators of the parameters of the following distributions:

i. Exponential distribution with parameter $\theta$,

ii. Normal distribution with mean $\mu$ and variance $\sigma^2$.

(b) Let $X_1, X_2, ..., X_n$ be $n$ a random sample from a normal distribution with unknown mean $\mu$ and known variance $\sigma^2$. Find $100(1 - \alpha)\%$ confidence interval for $\mu$.

(c) On the basis of results obtained from a random sample of 100 men from a particular district, the 95% confidence interval for the mean height of the men in the district is found to be (177.22 cm, 179.18 cm). Find the value of $\overline{X}$, the mean of the sample, and $\sigma^2$, the standard deviation of the normal population from which the sample is drawn. Calculate the 98% confidence interval for the mean height.