EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE 2003/2004

June/July 2005

SECOND SEMESTER

MT 301 - GROUP THEORY

REPEAT

Answer all questions

Time: Three hours

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1. (a) Define the following terms:

    i. Group and,

    ii. Subgroup of a group.

   (b) Let H be a non-empty subset of a group G. Prove that, H is a subgroup of G if and only if $ab^{-1} \in H,\ \forall\ a, b \in H$.

   (c) Let H be a subgroup of a group G. Prove that $H^{-1} = H$ and $H^n = H\ \ \forall n \in \mathbb{N}$.

   Is it true that, if $H^{-1} = H$ then H is a subgroup of G? Justify your answer.

   (d) Let H and K be two subgroups of a group G. Prove that

    i. $H \cup K$ need not be a subgroup of G, and

    ii. if $H \cup K$ is a subgroup of G, then $H \subseteq K$ or $K \subseteq H$.

    iii. Let $\{H_\alpha\}_{\alpha \in I}$ be an arbitrary family of subgroups of a group G, then prove that $\bigcap_{\alpha \in I} H_\alpha$ is a subgroup of G.

2. (a) State and prove Lagrange's theorem for a finite group G.

   (b) In a group G, H and K are different subgroups of order p, p is prime. Show that $H \cap K = \{e\}$, where e is the identity element of G.

   (c) Prove that in a finite group G, the order of each element divides order of G. Hence prove that $x^{|G|} = e, \quad \forall x \in G$.

   (d) Let G be a non-abelian group of order 10. Prove that G contains at least one element of order 5.

   (e) If every non-identity element of a group G has order 2, show that G is abelian.

3. (a) State and prove the first **isomorphism theorem.**

   (b) Let H be a subgroup of a group G and K be a normal subgroup of G. Prove with usual notations that,

      i. $K \trianglelefteq HK$.

      ii. $\dfrac{H}{H \cap K} \cong \dfrac{HK}{K}$.

4. What is meant by " **two elements are conjugate in a group** G "?

   (a) Let G be a group and $a, b \in G$. Define a relation " $\sim$ " on G by

   $$a \sim b \Leftrightarrow a \text{ and } b \text{ are conjugate in } G.$$

   Prove that " $\sim$ " is an equivalence relation on G.

   Given $a \in G$, let $\Gamma(a)$ denote the equivalence class containing a.

   Show that $|\Gamma(a)| = [G : C(a)]$, and $a \in Z(G) \Leftrightarrow \Gamma(a) = \{a\}$, where $C(a) = \{x \in G \ / ax = xa\}$ and $Z(G)$ is the center of the group G.

   (b) Write down the class equation of a finite group G.

   Hence or otherwise, prove that if the order of G is p^n, where p is a prime number and n is a positive integer then the center of G is non-trivial.

5. (a) Define the term "p-group".

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

(b) Let G' be the commutator subgroup of a group G. Prove the following:

i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G.

ii. G' is a normal subgroup of G.

iii. $\frac{G}{G'}$ is abelian.

6. Define the following terms:

* homomorphism
* isomorphism
* automorphism and inner automorphism.

(a) Prove the following:

i. homomorphic image of an abelian group is abelian.

ii. homomorphic image of a cyclic group is cyclic.

(b) Let **AutG** be the set of all automorphism of a group G and let **InnG** be the set of all inner automorphism of G. Show that,

i. **AutG** is a group under composition of maps.

ii. **InnG** is a normal subgroup of **AutG**.