EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE 2003/2004

June/July 2005

SECOND SEMESTER

MT 301 - GROUP THEORY

Answer all questions

Time: 3 hours

1. (a) Define the following terms:

   i. Group,

   ii. Subgroup of a group.

   (b) Let H be a non-empty subset of a group G. Prove that, H is a subgroup of G if and only if $ab^{-1} \in H, \ \forall\, a, b \in H$.

   (c) Let H be a subgroup of a group G. Prove that $H^{-1} = H$ and $H^n = H \ \ \forall n \in \mathbb{N}$.

   Is it true that, if $H^{-1} = H$ then H is a subgroup of G? Justify your answer.

   (d) Let H and K be two subgroups of a group G. Prove that

   i. $H \cup K$ need not be a subgroup of G, and

   ii. if $H \cup K$ is a subgroup of G, then $H \subseteq K$ or $K \subseteq H$.

   (e) Let $\{H_\alpha\}_{\alpha \in I}$ be an arbitrary family of subgroups of a group G, then prove that $\bigcap_{\alpha \in I} H_\alpha$ is a subgroup of G.

2. State and prove Lagrange's theorem for a finite group G.

(a) If every non-identity element of a group G has order 2, show that G is abelian.

(b) Let x and y be elements of a group G. Show that the element $x^{-1}yx$ has the same order as y.

(c) Let x and y be elements of a group, with order$(x) = 5$. Show that if x^3 and y commute then x and y commute.

(d) Let G be a non-abelian group of order 10. Prove that G contains at least one element of order 5.

3. (a) State and prove the **first isomorphism theorem**.

(b) Let H be a subgroup of a group G and K be a normal subgroup of G. Prove with usual notations that,

i. $K \trianglelefteq HK$.

ii. $\dfrac{H}{H \cap K} \cong \dfrac{HK}{K}$.

4. Explain what is meant by saying that "**two elements are conjugate in a group G**".

(a) Let G be a group and $a, b \in G$. Define a relation "$\sim$" on G by

$$a \sim b \Leftrightarrow a \text{ and } b \text{ are conjugate in } G.$$

Prove that "$\sim$" is an equivalence relation on G.

Given $a \in G$, let $\Gamma(a)$ denote the equivalence class containing a.

Show that $|\Gamma(a)| = [G : C(a)]$, and $a \in Z(G) \Leftrightarrow \Gamma(a) = \{a\}$, where $C(a) = \{x \in G \,|ax = xa\}$ and $Z(G)$ is the centre of the group G.

(b) Write down the class equation of a finite group G.

Hence or otherwise, prove that if the order of G is p^n, where p is a prime number and n is a positive integer, then the centre of G is non-trivial.

5. (a) Define the term "p-group".

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

(b) Define the term " internal direct product of groups ".

Give an example to show that some groups **can not** be expressed as the internal direct product of non-trivial normal subgroups of a group.

6. Define the following terms:

* homomorphism
* isomorphism
* automorphism and inner automorphism.

(a) Prove the following:

i. homomorphic image of an abelian group is abelian.

ii. homomorphic image of a cyclic group is cyclic.

(b) Let $\boldsymbol{AutG}$ be the set of all automorphisms of a group G and let $\boldsymbol{InnG}$ be the set of all inner automorphisms of G. Show that,

i. $\boldsymbol{AutG}$ is a group under composition of maps.

ii. $\boldsymbol{InnG}$ is a normal subgroup of $\boldsymbol{AutG}$.