EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE 2003/2004

Jun./Jul. 2005

SECOND SEMESTER

MT 303 - FUNCTIONAL ANALYSIS

Answer all questions

Time: 2 hours

1. Define the term " Banach space ".

   (a) Show that the sequence space

$$l^{\infty} = \{ \; x = (x_i) \; ; x_i \in \mathbb{C}, \; \sup_i |x_i| < \infty \; \}$$

   with the norm defined by $\|x\| = \sup_i |x_i|$ is a Banach space.

   (b) Show with the usual notation that $(e_i)_{i=1}^{\infty}$ is a Schauder basis for C_0, where

   $C_0 = \{x = (x_i) : x_i \in \mathbb{C}, \; (x_i) \text{ converges to zero} \}$ with the norm

$$\|x\| = \sup_{i \in \mathbb{N}} |x_i|.$$

2. Let $\{x_1, x_2, \ldots\ldots\ldots\ldots, x_n\}$ be a linearly independent set of vectors in a norm linear space X. Prove that there is a number $c > 0$ such that

$$\|\beta_1 x_1 + \beta_2 x_2 + \ldots\ldots\ldots + \beta_n x_n\| \geq c(|\beta_1| + |\beta_2| + \ldots\ldots\ldots + |\beta_n|)$$

   for every choice of scalars $\beta_1, \beta_2, \ldots\ldots, \beta_n$.

   Hence prove that a finite dimensional subspace Y of X is complete.

3. Prove or disprove the following;

(a) Let X and Y be two norm linear spaces and let $T : X \to Y$ be linear. T is continuous if and only if T is bounded.

(b) Every linear operator on a norm linear space is bounded.

(c) The dual space of l^1 is l^∞.

4. State the Hahn Banach theorem for norm linear spaces.

(a) Let X be a norm linear space and let $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional f^* on X such that

$$\|f^*\| = 1 \text{ and } f^*(x_0) = \|x_0\| \text{ and}$$

prove that

if $f(x) = f(y)$ for every bounded linear functional on X then $x = y$.

(b) Let Y be a proper closed subspace of a norm linear space X.

Let $x_0 \in X \backslash Y$ and $\delta = \inf_{y \in Y} \|y - x_0\|$. Show that there exists a bounded linear functional F on X such that $\|F\| = 1$, $F(y) = 0 \ \ \forall y \in Y$ and $F(x_0) = \delta$.