# EASTERN UNIVERSITY, SRI LANKA

## THIRD EXAMINATION IN SCIENCE - 2003/2004

## SECOND SEMESTER (Apr.'2006)

## MT 303 - FUNCTIONAL ANALYSIS

## REPEAT

Answer all questions — Time allowed: Two hours

1. Define the term " Banach space ".

   (a) Show that the sequence space

   $$l^{\infty} = \{ \ x = (x_i) \ ; x_i \in \mathbb{C}, \ \sup_{i \in \mathbb{N}} |x_i| < \infty \ \}$$

   with the norm defined by $\|x\| = \sup_{i \in \mathbb{N}} |x_i|$ is a Banach space.

   (b) Show with the usual notation that $(e_i)_{i=1}^{\infty}$ is a Schauder basis for $C_0$, where

   $C_0 = \{x = (x_i) : x_i \in \mathbb{C}, \ (x_i) \, \text{converges to zero} \}$ with the norm

   $$\|x\| = \sup_{i \in \mathbb{N}} |x_i|.$$

2. If $\{x_1, x_2, ..., x_n\}$ is a linearly independent set of vectors in a normed linear space $X$, there is a number $c > 0$ such that

$$\|\beta_1 x_1 + \beta_2 x_2 + .... + \beta_n x_n\| \geq c(|\beta_1| + |\beta_2| + ... + |\beta_n|)$$

for every choice of scalars $\beta_1, \beta_2, ..., \beta_n$. Use this result to prove the following:

(a) prove that every finite dimensional subspace $Y$ of $X$ is complete and closed,

(b) prove that any two norms on a finite dimensional linear space are equivalent.

3. Define the term "bounded linear operator" from a normed linear space into another normed linear space.

(a) Let $T : X \to Y$ be a linear operator, where $X$ and $Y$ are normed linear spaces. Prove that $T$ is continuous if and only if $T$ is bounded.

(b) If $T$ is a linear operator from a normed linear space $X$ onto a normed linear space $Y$, then show that the inverse operator $T^{-1} : Y \to X$ exists and is bounded linear if and only if there exists $k > 0$ such that

$$\|Tx\| \geq k\|x\| \text{ forall } x \in X.$$

4. State the Hahn Banach theorem for normed linear spaces.

(a) Let $X$ be a normed linear space and let $x_0 \neq 0$ be any element of $X$. Prove that there exists a bounded linear functional $f^*$ on $X$ such that

$$\|f^*\| = 1 \text{ and } f^*(x_0) = \|x_0\|.$$

Further, Prove that, if $f(x) = f(y)$ for every bounded linear functional $f$ on $X$ then $x = y$.

(b) Let $Y$ be a proper closed subspace of a norm linear space $X$.

Let $x_0 \in X \backslash Y$ and $\delta = \inf_{y \in Y} \|y - x_0\|$. Show that there exists a bounded linear function $F$ on $X$ such that $\|F\| = 1$ , $F(y) = 0 \quad \forall y \in Y$ and $F(x_0) = \delta$.