EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2004/2005)

FIRST SEMESTER (Jan./Feb., 2006)

MT 306 - PROBABILITY THEORY

Answer all questions | Time allowed: Two hours

1. (a) Define the term "conditional probability".

Let A and B be two events.

i. Show that

$$P(A|B') = \frac{P(A) - P(A \cap B)}{1 - P(B)}.$$

Hence show that $P(A \cap B) \geq P(A) + P(B) - 1$.

ii. If $P(A) > P(B)$, then show that $P(A|B) > P(B|A)$.

(b) A random variable X has Poisson distribution with parameter λ given by

$$P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!} \;\; ; \;\; x = 0, 1, 2, \cdots.$$

Find the mean, variance and moment generating function of X.

(c) During office hours, telephone calls to a telephone in an office come in at an average rate of 20 calls per hour. Assuming that a Poisson distribution is relevant, write down the probability function of X, the number of telephone calls arriving in each five-minute period. Find the probability that there will be

i. less than two calls,

ii. more than three calls,

in a five minute period.

2. (a) If X is a random variable with density function f_X and $g(x)$ is monotonically increasing and differentiable function from $\mathbb{R}$ into $\mathbb{R}$, show that $Y = g(X)$ has the density function

$$f_Y(g) = f_X[g^{-1}(y)]\ \frac{d}{dy}[g^{-1}(y)], \quad y \in \mathbb{R}.$$

If the probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{kx^3}{(1+2x)^6} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant, find the probability density function of the random variable $Y = \dfrac{2X}{1+2X}$.

(b) Random variables X and Y have joint density function

$$f_{XY}(x,y) = \begin{cases} c(x^2 + \frac{1}{2}xy) & \text{if } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find

i. the value of c,

ii. marginal density functions of X and Y,

iii. $E(XY)$.

3. (a) Define the following terms:

i. Unbiased estimator;

ii. Risk function.

(b) Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution with mean μ and variance θ.

i. Show that

$$\sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\theta} \sim \psi^2_{n-1}.$$

Hence, show that $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is an unbiased estimator of θ.

ii. Find the risk function of S^2.

iii. Let $T = bY$ be an estimator for θ, where $Y = \frac{(n-1)S^2}{n}$. Show that the risk function of T is

$$\frac{\theta^2}{n^2}\left[(n^2-1)b^2 - 2n(n-1)b + n^2\right].$$

Deduce that $b = \frac{n}{n+1}$, if risk is minimum.

4. (a) Define the maximum likelihood estimator.

Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. Find the maximum likelihood estimators for μ and σ^2.

(b) Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution with unknown mean μ_1 and known variance σ_1^2 and let $Y_1, Y_2, \cdots, Y_m$ be a random sample from a normal distribution with unknown mean μ_2 and known variance σ_2^2. Find the $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$.

(c) Suppose we wish to measure the difference in sales between two types of employees in the insurance industry. One type consists of people who are college graduates, and the other consists of high school graduates only. A random sample of 45 employees is taken from those with a college degree and the mean of monthly sales is 32(in thousand dollars), while the mean of a sample of 60 employees with high school diplomas is 25. It is assumed that the variance of the monthly sales is known to be 48 for college graduates and 56 for high school graduates. Find the 95% confidence interval for the difference between the average sales of the two types of salesmen.