EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE 2004/2005

FIRST SEMESTER (MARCH / APRIL' 2006)

(Repeat)

MT 202 - METRIC SPACE

Answer all questions — Time allowed: Two hours

---

1. Define the term *complete metric space*.

   (a) Prove that $\mathbb{R}$ with usual metric is complete.

   (b) Define the metric d on $\mathbb{R}^n$ by

   $$d(\underline{x}, \underline{y}) = \left(\sum_{i=1}^{n} | x_i - y_i |^2 \right)^{\frac{1}{2}}, \quad \forall\ \underline{x} = (x_1, x_2, \cdots, x_n), \underline{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$$

   Prove that $(\mathbb{R}^n, d)$ is a complete metric space.

2. (a) Let A be a subset of a metric space (X, d). Define the term *interior of A*.

   Prove that A^o, the interior of A, is the largest open set contained in A.

   (b) Let A, B be any two subsets of a metric space (X, d). Prove that:

   i. $A^o \cap B^o = (A \cap B)^o$,

   ii. $A^o \cup B^o \subseteq (A \cup B)^o$.

   Give an example to show $A^o \cup B^o \neq (A \cup B)^o$.

3. Define the terms *compact set* and *separated sets.*

(a) Let A and B be two subsets in a metric space (X, d). Prove the following:

i. If $d(A, B) > 0$ then A and B are separated, where $d(A, B)$ denotes the distance between A and B.

ii. If A and B are separated and $A \cup B$ is open then A and B are open.

iii. If A and B are connected and not separated then their union is connected.

(b) Prove that the closed interval $[a, b]$ is compact under the $\mathbb{R}$ with usual metric.

4. Let (X, d_X) and (Y, d_Y) are two metric spaces and f be a function from X to Y. Prove the following:

(a) f is continuous at a iff every sequence $\{a_n\}$ in X converging to a implies $\{f(a_n)\}$ converging to $f(a)$.

(b) f is continuous iff $f^{-1}(G)$ is open in X whenever G is open in Y.

(c) f is continuous iff $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ \quad \forall\, B \subseteq Y$.