PER ARDUA AD SCIENTIAM

EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (Sept./Oct., 2010)

MT 202 - ANALYSIS II(METRIC SPACE)

Answer all questions — Time: Two hours

1. (a) Define the term *metric space*.

   Let (X, d) be a metric space. Define a mapping $d' : X \times X \longrightarrow \mathbb{R}$ by
   $$d'(x, y) = \min(1, d(x, y)).$$
   Show that d' is a metric on X.

   (b) Let (X, d) be a metric space and let (a_n), (b_n) be sequence in X. Prove that

   i. if $a_n \longrightarrow a$ and $b_n \longrightarrow b$ as $n \to \infty$ then $d(a_n, b_n) \longrightarrow d(a, b)$ as $n \to \infty$;

   ii. if (a_n) is convergent then it is bounded.

2. (a) Let A be a subset of a metric space (X, d). Define the followings:

   i. Closure of A;

   ii. Subspace of (X, d).

   (b) Let (X, d) be a complete metric space and Y be a subspace of X. Prove that Y is complete if and only if Y is closed in (X, d).

   (c) Prove that if A is a subset of metric space (X, d) then closure of A is the smallest closed set contains A.

   (d) Prove that in any metric space, singleton sets are closed sets.

3. (a) Define the following terms in a metric space:

i. Separated set;

ii. Disconnected set.

(b) Let Y be a subset of a metric space (X, d). Prove that the following statements are equivalent.

i. Y is connected;

ii. Y cannot be expressed as a disjoint union of two non-empty closed sets in Y;

iii. Φ and Y are the only sets which are both open and closed in Y.

4. (a) What is meant by a function from a metric space (X, d) to a metric space (Y, ρ) is continuous at $a \in X$?

i. Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a mapping $f : X \longrightarrow Y$ is continuous on X if and only if for all closed sets of G in Y, the set $f^{-1}(G)$ is closed in X.

ii. Let $X = C[0,1]$, the set of all continuous functions on $[0,1]$, and let d be a metric on X defined by $d(f,g) = \int_0^1 |f(x) - g(x)|\,dx$. If $f_n(x) = x^n$ for all $x \in [0,1]$ and $n \in \mathbb{N}$ then show that the sequence (f_n) converges in X.

(b) Define the term *compact set*.
Prove that the set $[0,1] \subset \mathbb{R}$ is compact.