EASTERN UNIVERSITY, SRI LANKA
SECOND EXAMINATION IN SCIENCE - 2007/2008
FIRST SEMESTER(Dec./Jan.,2008/2009)
MT 207 - NUMERICAL ANALYSIS
(PROPER & REPEAT)

Answer all Questions **Calculators are provided** **Time: Two hours**

Q1. (a) Define the term "relative round - off error" in the floating point number system. Show that

$$|\delta x| \leq \frac{1}{2}\beta^{1-t},$$

where δx represents the relative round - off error obtained by rounding a number in the floating point number system, is held for a β-base computer with t digits in the mantissa.

Hence, find the absolute value of the error bound for a decimal computer with ten digits in the mantissa.

(b) Compare the results of $P(2.19)$ and $Q(2.19)$ using three - digit rounding arithmetic, where

$$\begin{aligned} P(x) &= ((x^3 - 3x^2) + 3x) - 1, \\ Q(x) &= ((x - 3)x + 3)x - 1, \end{aligned}$$

and the true values of $P(2.19)$ and $Q(2.19)$ are the same and it is 1.685159. State the significance of this problem.

Q2. Let $x = g(x)$ be an arrangement of the equation $f(x) = 0$, which has a root α in the interval I. If $g'(x)$ exists and continuous in I satisfying

$$|g'(x)| \leq h < 1, \ \forall x \in I,$$

prove that, for any given x_0, the sequence $\{x_r\}$, $r = 0, 1, 2, \ldots$, defined by

$$x_{r+1} = g(x_r)$$

converges to the root α and such α is unique.

Hence, find the condition for the convergence of Newton-Raphson method.

Furthermore, show that the convergence of the method described above depends on the choice of $g(x)$ by considering the equation

$$f(x) = x - e^{-x}$$

with an approximate root 0.567.

Q3. (a) If $f \in C^{n+1}[a,b]$ and p_n is the Lagrange's interpolating polynomial which interpolates the function $f(x)$ at the distinct points $x_0, x_1, \ldots x_n$ in [a,b], prove that for all $x \in [a,b]$, there exists $\xi \in (a,b)$ such that the error, $R_n(x)$, in interpolation is

$$R_n(x) = \frac{\prod_{n+1}(x)}{(n+1)!} f^{n+1}(\xi),$$

where $a < \xi < b$ and $\prod_{n+1}(x) = (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_n)$.

Hence, if the function $f(x) = \ln x$ is interpolated by a polynomial of degree two, in the interval [2,3] with step length 0.5, show that

$$| R_2(x) | \leq 0.00175 ,$$

at $x = 2.7$

(b) With the usual notations, write down the formulas for finding the integral

$$I = \int_{x_0}^{x_n} f(x)\, dx,$$

truncation error and round-off error for the composite Trapezoidal rule.

Calculate a value for the step length h which will ensure that the truncation error in the composite Trapezoidal rule approximation to the integral

$$I = \int_1^2 \frac{1}{x}\, dx$$

is at most 10^{-2} in absolute value. Further, find the round-off error and hence compute an estimation of I which is correct to this accuracy.

Q4. (a) What is meant by a linear system of equations $A\underline{x} = \underline{b}$ is "ill-conditioned" and "well-conditioned"?

(b) By finding the condition number, check whether the system given below is ill-conditioned

$$\begin{aligned} -0.6x_1 + 0.6x_2 &= 1.25, \\ 0.4x_1 + 0.2x_2 &= 2.75. \end{aligned}$$

You may use an Euclidean norm which is defined by $\|.\|_2 = \sqrt{\sum_{ij} |a_{ij}|^2}$.

(c) Consider the following system of linear equations:

$$\begin{aligned} 83x_1 + 11x_2 - 4x_3 &= 95, \\ 7x_1 + 52x_2 + 13x_3 &= 104, \\ 3x_1 + 8x_2 + 29x_3 &= 71. \end{aligned}$$

Is this system convergent for Gauss-Seidel method? If so, apply this method to carry out three iterations for x_1, x_2 and x_3 correct to four decimal places.