EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2004/2005

SECOND SEMESTER (Oct./ Nov., 2006)

MT 301 - GROUP THEORY

(Proper and Repeat)

Answer all questions — Time : Three hours

1. (a) Define the following terms:

   i. group,

   ii. subgroup of a group.

   (b) Let H be an non-empty subset of a group G. Prove that, H is a subgroup of G if and only if $ab^{-1} \in H, \ \forall \ a, b \in H$.

   (c) Let G be a group and H be a non-empty subset of G. Prove that H is a subgroup of G if and only if $HH^{-1} = H$.

   (d) Let H and K be two subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$.

2. (a) State and prove **Lagrange's theorem** for a finite group G.

   (b) Prove that in a finite group G, the order of each element divides the order of G. Hence prove that every group of prime order is cyclic; moreover, in a group of prime order p, every non-identity element has order p.

   (c) Let p and q be two distinct prime numbers and let G be a group of order pq. Show that every proper subgroup of G is cyclic.

   (d) Let G be a non-abelian group of order 8. Prove that G contains at least one element of order 4.

3. (a) State and prove the **first isomorphism theorem**.

   (b) Let G be a group such that for fixed integer $n > 1$, $(ab)^n = a^n b^n$ for all $a, b \in G$. Let $G_n = \{a \in G : a^n = e\}$ and $G^n = \{a^n : a \in G\}$. Prove that $G_n \trianglelefteq G$ and $\dfrac{G}{G_n} \cong G^n$.

4. Prove or disprove the following:

   (a) If H and K are two subgroups of a group G then $H \cup K$ is a subgroup of G.

   (b) The homomorphic image of an abelian group is abelian.

   (c) If all non-trivial subgroups of a group G are cyclic then G is cyclic.

   (d) Any group of order p^n, where p is prime, has non-trivial center.

5. (a) Define the term p-group.

   Prove the following:

   i. Every subgroup of a p-group is a p-group.

   ii. The homomorphic image of a p-group is a p-group.

   (b) Define the commutator subgroup G' of a group G.

   Prove the following:

   i. $G' \trianglelefteq G$,

   ii. $\frac{G}{G'}$ is abelian,

   iii. If $H \trianglelefteq G$; then $\frac{G}{H}$ is abelian if and only if $G' \subseteq H$.

6. Define the following terms as applied to a group:

* Permutation,

* Cycle of order r,

* Transposition,

* Signature.

   (a) Prove that the set S_n of all permutations on n symbols forms a group.

   (b) Prove that the permutation group on n symbols (S_n) is a finite group of order $n!$.

   Is it true that S_n is abelian for $n > 2$? Justify your answer.

   (c) Express the permutation,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

   as a product of disjoint cycles. Hence or otherwise, find the inverse of σ and determine whether σ is even or odd.