EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE(2004/05)

(Oct./Nov.'2006)

SECOND SEMESTER

MT 303 - FUNCTIONAL ANALYSIS

(Proper & Repeat)

Answer all questions **Time: Two hours**

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1. Define the term "Banach space".

(a) (i) Let $0 < p < 1$ and $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$, $x \in \mathbb{C}^n$. Show that $\|.\|_p$ is not a norm on $\mathbb{C}^n$, if $n \geq 2$.

(ii) If $1 \leq p < \infty$, prove that $\|.\|_p$ is a norm on $\mathbb{C}^n$ and

$$\mathbb{C}^n = \left\{ x = (x_i) : x_i \in \mathbb{C}, \sum_{i=1}^{n} |x_i|^p < \infty \right\}$$

is a Banach space with this norm.

Also prove the following inequalities;

for any $x \in \mathbb{C}^n$,

(A) $n^{-\frac{1}{2}} \| x \|_2 \leq \| x \|_p \leq \| x \|_2$, if $p > 2$

(B) $\| x \|_2 \leq \| x \|_p \leq n^{\frac{1}{2}} \| x \|_2$, if $1 \leq p \leq 2$

(Hint:You may assume $\left(\sum_{i=1}^{n} |a_i|\right)^p \geq \sum_{i=1}^{n} |a_i|^p$, for any $a = (a_i) \in \mathbb{C}^n$, if $p \geq 1$.)

(b) Let X and Y be Banach spaces. Verify whether the product space $X \times Y$ with the norm defined by

$$\| (x, y) \| = \| x \| + \| y \|, \quad \forall\, (x, y) \in X \times Y$$

is a Banach space.

2. (a) If $\{x_1, x_2, \cdots, x_n\}$ is a set of linearly independent vectors in a normed linear space X, then there exists a number $k > 0$ such that

$$\| \sum_{i=1}^{n} \eta_i x_i \| \geq k \sum_{i=1}^{n} |\eta_i|$$

for every choice of scalars $\eta_1, \eta_2, \cdots, \eta_n$. Use this result to prove the following:

i. Every finite dimensional subspace of X is complete.

ii. Any two norms on a finite dimensional normed linear space are equivalent.

(b) Show that two norms on a linear space are equivalent if and only if every Cauchy sequence with respect to one of the norms is a Cauchy sequence with respect to other norm.

(c) Prove that a normed linear space X is of finite dimension if and only if the unit ball $\{x \in X : \| x \| \leq 1\}$ is compact.

(Hint: You may use Riesz's lemma)

3. Define the term "bounded linear operator" from a normed linear space into another normed linear space.

(a) Let $X = \mathbb{R}^2$ with the Euclidean norm $\| \cdot \|_2$ and T be a bounded linear operator from X to itself represented by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that $\| T \| = \frac{1}{2}\left(\sqrt{\alpha} + \sqrt{\beta}\right)$, where $\alpha = a^2 + b^2 + c^2 + 2(ad - bc)$, and $\beta = a^2 + b^2 + c^2 - 2(ad - bc)$.

(b) If T is a linear operator from a normed linear space X onto a normed linear space Y, then show that the inverse operator $T^{-1} : Y \to X$ exists and is bounded if and only if there exists $k > 0$ such that

$$\| T(x) \| \geq k \| x \|, \quad \text{for all } x \in X.$$

(c) Let X be the normed linear space of all bounded real valued functions on $\mathbb{R}$ with norm defined by

$$\| x \| = \sup \{ | x(t) | : t \in \mathbb{R} \}, \ \forall \, x \in X.$$

Let $T : X \to Y$ be defined by $T(x(t)) = x(t - \tau)$, where $\tau > 0$ is a constant. Is T linear? Bounded?

4. State the Hahn Banach theorem for normed linear spaces.

(a) Let Y be a proper closed subspace of a normed linear space X and let $x_0 \in X \setminus Y$. Show that there exists a bounded linear functional f_0 defined on X such that $f_0(Y) = 0$ and $f_0(x_0) \neq 0$.

(b) Let X be a normed linear space and let $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional g on X such that $\| g \| = 1$ and $g(x_0) = \| x_0 \|$.

Deduce that if $f(x) = f(y)$ for every bounded linear functional on X then $x = y$.