EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE 2004/2005

FIRST SEMESTER(Nov./Dec.'2006)

SPECIAL REPEAT EXAMINATION

MT 306 - PROBABILITY THEORY

Answer all questions **Time: Two hours**

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1. (a) Define the term "probability space".

   i. Show that the probability that exactly one of the events A or B occurs is

$$P(A) + P(B) - 2P(A \cap B) .$$

   ii. Prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

   where A_i $(i = 1, 2, \cdots, n)$ are events defined on the sample space.

   (b) i. A random variable X has Poisson distribution with parameter λ. Find the mean and variance of X.

   ii. A book contains 100 typing errors distributed randomly throughout in 500 pages. Find the probability that a page selected at randomly contains

      A. at least two errors.

      B. 10 pages selected at random contain 5 or more errors.

2. Define the "Moment generating function" of a random variable X.

(a) Find the moment generating function for the Gamma distribution is given by

$$f_X(x) = \begin{cases} \dfrac{\lambda^m x^{m-1} e^{-\lambda x}}{\Gamma(m)} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence find the mean and variance.

(b) Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that the random variable $Y = -2\log X$ has a chi-square distribution with 2 degrees of freedom.

(c) The random variable X follows the normal distribution with mean 0 and variance 1. Find the moment generating function of X^2.

Deduce that, if $X_1, X_2, \cdots, X_n$ are independent random variables having the normal distribution with mean 0 and variance 1, then $U = X_1^2 + X_2^2 + \cdots + X_n^2$ has a Chi-square distribution with n degrees of freedom.

3. (a) Define the following terms:

i. Unbiased estimator;

ii. Risk function.

(b) Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution with mean μ and variance θ.

i. Show that
$$\sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\theta} \sim \psi^2_{n-1}\ .$$

Hence, show that $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is an unbiased estimator of θ.

ii. Find the risk function of S^2.

iii. Let $T = bY$ be an estimator for θ, where $Y = \dfrac{(n-1)S^2}{n}$. Show that the risk function of T is

$$\frac{\theta^2}{n^2}\left[(n^2-1)b^2 - 2n(n-1)b + n^2\right] .$$

Deduce that $b = \dfrac{n}{n+1}$, if risk is minimum.

4. (a) Define the maximum likelihood estimatior.

Let $X_1,\ X_2, \cdots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. Find the maximum likelihood estimator for μ and σ^2.

(b) Let us assume that the sample data follow a normal distribution with unknown mean μ and unknown variance σ^2. If $n = 12$, $\sum_{i=1}^{12} x_i = 180$ and $\sum_{i=1}^{12} x_i^2 = 2799$, find the maximum likelihood estimators for μ and σ^2.

(c) A machine which packs sugar has, for a long time, given a normal distribution of weights of filled packets, and the standard deviation of weights has been 2.5 grams. It is adjusted to give a new metric size of pack and 20 of the new packets are weighed. Their mean is 1002 grams. Find the 95% confidence limits for the true mean weight after the adjustment.