# EASTERN UNIVERSITY, SRI LANKA

## SECOND EXAMINATION IN SCIENCE 2005/2006

## March/April' 2008

## SECOND SEMESTER

## MT 202 - METRIC SPACE

## Proper & Repeat

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Answer all questions                                                    Time:Two hours

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Q1. Define the term *Complete Metric Space.*

(a) Show that the space $C_{[a,b]}$, the set of all real valued continuous functions on the interval $[a, b]$, with the function $d : C_{[a,b]} \times C_{[a,b]} \longrightarrow \mathbb{R}$ defined by

$$d(f, g) = \left( \int_a^b (f(x) - g(x))^2 dx \right)^{\frac{1}{2}} \qquad \text{for} \quad f, g \in [a, b],$$

is a metric space and also show that it is not a complete metric space.

(b) Prove that a subset $A$ of $X$ is nowhere dense in $X$ if and only if $(\overline{A})^\circ = \emptyset$.

Q2. (a) Let $(X, d)$ be any metric space and let $A$ be any non empty subset of $X$. Prove that $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ in $A$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

(b) Prove that separated sets are disjoint. Is the converse part true? Justify your answer.

(c) Prove that a metric space $(X, d)$ is disconnected if and only if it can be written as a union of two non empty disjoint open sets.

(d) Prove that a metric space $(X, d)$ is disconnected if and only if there exists a non empty proper subset of $X$ which is both open and closed.

Q3. Define the term *Compact Set*.

(a) Show that $[a, b]$ is compact in $(\mathbb{R}, |.|)$ and also show that $(0, 1)$ is not compact in $(\mathbb{R}, |.|)$.

(b) Prove that continuous image of a compact subset is compact.

(c) If $A$ is compact subset of $(X, d)$. Prove that $A$ is closed.

Q4. Define the term *Continuous Function* between two metric spaces.

(a) Let $(X, d_1)$ and $(Y, d_2)$ be any two metric spaces and $f : X \longrightarrow Y$ be a function. Prove that $f$ is continuous at $a \in X$ if and only if for every sequence $\{a_n\}$ in $X$ converging to $a$ we have $\{f(a_n)\}$ converging to $f(a)$.

(b) Let $(X, d_1)$ and $(Y, d_2)$ be any two metric spaces and $f : X \longrightarrow Y$ be a function. Prove that $f$ is continuous if and only if $f^{-1}(G)$ is open in X, whenever $G$ is open in $Y$.

(c) Prove that $f : X \longrightarrow Y$ is continuous if and only if $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$

(d) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0,0); \\ 0, & \text{if } (x, y) = (0,0). \end{cases}$$

where $\mathbb{R}^2$ and $\mathbb{R}$ are considered with respect to the usual metric. Discuss the continuity at the origin.