EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (Sep./Nov., 2010)

MT301 - GROUP THEORY

(PROPER & REPEAT)

Answer all questions **Time : Three hours**

1. (a) Define the term *group*.

   (b) Let p be a fixed positive prime and $G = \{1, 2, ..., p-1\}$. If the binary operation of multiplication modulo p, denoted by $\odot_p$, is defined on G, show that $(G, \odot_p)$ is a group

   (c) i. Let H be a non-empty subset of a group G. Prove that, H is a subgroup of G if and only if $ab^{-1} \in H, \ \forall\, a, b \in H$.

   ii. Let H and K be two subgroups of a group G. Is $H \cup K$ a subgroup of G? Justify your answer.

   iii. Let $\{H_\alpha\}_{\alpha \in I}$ be an arbitrary family of subgroups of a group G. Prove that $\bigcap_{\alpha \in I} H_\alpha$ is a subgroup of G.

2. (a) State and prove the *Lagrange's theorem* for a finite group G.

Let G be a group and let H and K be subgroups of G such that $|H| = 12$ and $|K| = 5$. Prove that $H \cap K = \{e\}$, where e is the identity element of G.

(b) Let G' be the commutator subgroup of G. Prove the followings:

i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G.

ii. $G' \trianglelefteq G$.

iii. Let F be the group of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, where $ad \neq 0$, under matrix multiplication. Show that F', the commutator subgroup of F, precisely the set of all matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$.

3. (a) State and prove the *first isomorphism theorem*.

(b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove that

i. $K \trianglelefteq H$;

ii. $H/K \trianglelefteq G/K$;

iii. $\dfrac{G/K}{H/K} \cong G/H$.

4. (a) Let G be a group and $g_1,\ g_2 \in G$. Define a relation " $\sim$ " on G by

$$g_1 \sim g_2 \Leftrightarrow \exists g \in G \text{ such that } g_2 = g^{-1}\, g_1\, g.$$

Prove that " $\sim$ " is an equivalence relation on G.

Given $a \in G$, let $\Gamma(a)$ be denote the equivalence class containing a. Show that:

i. $|\Gamma(a)| = |G : C(a)|$, where $C(a) = \{x \in G / ax = xa\}$;

ii. $a \in Z(G) \quad \Leftrightarrow \quad \Gamma(a) = \{a\}$, where $Z(G)$ is the center of the group G.

(b) Write down the class equation of a finite group G. Hence or otherwise, prove that the center of G is non-trival if the order of G is p^n, where p is a positive prime number.

5. (a) Define the term *p-group*.

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

(b) Define the term *homomorphism*.

Let G be the group of all real 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $ad - bc \neq 0$, under matrix multiplication. Let $\overline{G}$ be the group of all non-zero real numbers under multiplication. Define a mapping

$$\phi : G \to \overline{G} \text{ by } \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

Prove that ϕ is a homomorphism of G onto $\overline{G}$.

6. (a) Define the following terms as applied to a group:

i. *permutation*;

ii. *cycle of order r*.

(b) Prove that the permutation group on n symbols ,S_n, is a finite group of order $n!$.

Is S_n abelian for $n > 2$? Justify your answer.

(c) Prove that the set of even permutations A_n forms a normal subgroup of S_n . Hence show that $\frac{S_n}{A_n}$ is a cyclic group of order 2.

(d) Express the permutation σ in S_8 as a product of disjoint cycles, where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 4 & 2 & 8 & 1 & 6 \end{pmatrix}.$$