EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2005/2006

SECOND SEMESTER (Sep.'2007)

MT 301 - GROUP THEORY

REPEAT

Answer all questions   Time allowed: Three hours

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1. (a) Define the following terms:

   i. group,

   ii. subgroup of a group.

   (b) Let H be a non-empty subset of a group G. Prove that, H is a subgroup of G if and only if $ab^{-1} \in H, \ \forall \ a, b \in H$.

   (c) Let H and K be subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$.

   (d) Let H and K be two subgroups of a group G. Is it true that $H \cup K$ a subgroup of G? Justify your answer.

   (e) Let $\{H_\alpha\}_{\alpha \in I}$ be an arbitrary family of subgroups of a group G. Prove that $\bigcap_{\alpha \in I} H_\alpha$ a subgroup of G.

2. (a) State and prove **Lagrange's theorem** for a finite group G.

(b) Prove that in a finite group G, the order of each element divides order of G. prove that every group of prime order is cyclic, moreover, in a group of prime p, every non-identity element has order p.

(c) Let p and q be two distinct prime numbers and let G be a group of order pq. that every proper subgroup of G is cyclic.

(d) Let G be a non-cyclic group of order 8. Show that $a^4 = e$ for every $a \in G$, w is the identity element of G.

3. (a) State and prove the **first isomorphism theorem** .

(b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove

i. $K \trianglelefteq H$;

ii. $H/K \trianglelefteq G/K$;

iii. $\dfrac{G/K}{H/K} \cong G/H$.

4. Prove or disprove the following:

(a) Let G be a group and $Z(G)$ be the centre of G. If $G/Z(G)$ is cyclic then G is abe

(b) If G is a finite group then $O(ab) = O(ba)$ for all $a, b \in G$. ($O(x)$ stands for the order of the element x.)

(c) Every abelian group is cyclic.

(d) Let $\Phi : G \to G_1$ be a homomorphism, where G and G_1 are two groups. If H normal subgroup of G then $\Phi(H)$ is a normal subgroup of G_1.

(e) Homomorphic image of a p-group is p-group.

5. (a) Define the term "p-group".

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

(b) Let G' be the commutator subgroup of a group G. Prove the following:

i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G.

ii. G' is a normal subgroup of G.

iii. G/G' is abelian.

6. Define the following terms:

* homomorphism;
* isomorphism;
* automorphism and inner automorphism.

(a) Prove the following:

i. homomorphic image of an abelian group is abelian.

ii. homomorphic image of a cyclic group is cyclic.

(b) Let $\boldsymbol{AutG}$ be the set of all automorphisms of a group G and let $\boldsymbol{InnG}$ be the set of all inner automorphisms of G. Show that,

i. $\boldsymbol{AutG}$ is a group under composition of maps.

ii. $\boldsymbol{InnG}$ is a normal subgroup of $\boldsymbol{AutG}$.