EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2005/2006

SECOND SEMESTER (Sep.'2007)

MT 303 - FUNCTIONAL ANALYSIS I

REPEAT

Answer all questions

Time allowed: Two hours

Q1. Define the term "Banach space".

(a) i. Let $0 < p < 1$ and $||x_p|| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$, $x \in \mathbb{C}^n$. Show that $||.||_p$ is not a norm on $\mathbb{C}^n$, if $n \geq 2$.

ii. If $1 \leq p < \infty$, prove that $||.||_p$ is a norm on $\mathbb{C}^n$ and

$$\mathbb{C}^n = \{x = (x_i) : x_i \in \mathbb{C}, \sum_{i=1}^{n} |x_i|^p < \infty\}$$

is a Banach space with this norm.

(b) Let X and Y be Banach spaces. Verify whether the product space $X \times Y$ with the norm defined by

$$||(x, y)|| = ||x|| + ||y||, \ \forall (x, y) \in X \times Y$$

is a Banach space.

Q2. (a) If $x_1, x_2, ..., x_n$ is a set of linearly independent vectors in a norm linear spac
then there exists a number $k > 0$ such that

$$||\sum_{i=1}^{n} \eta_i x_i|| \geq k \sum_{i=1}^{n} |\eta_i|$$

for every choice of scalars $\eta_1, \eta_2, ..., \eta_n$. Use this result to prove the followin

i. Every finite dimensional subspace of X is complete;

ii. Any two norms on a finite dimensional normed linear space are equivalent.

(b) Prove that a normed linear space X is of finite dimension if and only if the ball $\{x \in X : ||x|| \leq 1\}$ is compact.

Q3. Define the term "bounded linear operator".

(a) Let $T : X \to Y$ be a linear operator, where X and Y are normed linear s
Prove that T is continuous if and only if T is bounded.

(b) If T is a linear operator from a normed linear space X onto a normed linear
Y, then show that the inverse operator $T^{-1} : Y \to X$ exists and is boun
and only if there exists $k > 0$ such that

$$||T(x)|| \geq k||x|| \quad \text{for all} \quad x \in X.$$

(c) Let X be the normed linear space of all bounded real valued functions on $\mathbb{R}$
norm defined by

$$||x|| = \sup\{|x(t)| : t \in \mathbb{R}\}, \quad \forall x \in X.$$

Let $T : X \to Y$ be defined by $T(x(t)) = x(t - r)$, where $r > 0$ is a constant
linear and bounded?

Q4. State the Hahn Banach theorem for normed linear spaces.

(a) Let X be a normed linear space and let $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional f^* on X such that

$$\|f^*\| = 1 \text{ and } f^*(x_0) = \|x_0\|.$$

Further, Prove that, if $f(x) = f(y)$ for every bounded linear functional f on X then $x = y$.

(b) Let Y be a proper closed subspace of a norm linear space X.

Let $x_0 \in X \backslash Y$ and $\delta = \inf_{y \in Y} \|y - x_0\|$. Show that there exists a bounded linear function F on X such that $\|F\| = 1$, $F(y) = 0 \ \forall y \in Y$ and $F(x_0) = \delta$.