EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2005/2006

FIRST SEMESTER (Aug./Sep.,2007)

MT 306 - PROBABILITY THEORY

(Proper & Repeat)

Answer all questions Time : Two hours

Q1. (a) i. State and prove the Baye's theorem.

ii. In a certain college, 4% of the men and 1% of the women are taller than 1.8 m. Furthermore 60% of the students are women. If a student selected at random is taller than 1.8 m, what is the probability that the student is a woman?

(b) A random variable X has Poisson distribution with parameter λ given by

$$P[X = x] = \frac{e^{-\lambda}\,\lambda^x}{x!}.$$

Find the mean, variance and the moment generating function of X.

(c) The mean number of bacteria per milliliter of a liquid is known to be 4. Assuming that the number of bacteria follows a Poisson distribution, find the probability that

i. in 1 ml of liquid there will be no bacteria,

ii. in 3 ml of liquid there will be less than two bacteria,

iii. in $\frac{1}{2}$ ml of liquid there will be more than two bacteria.

Q2. (a) If X is a random variable with density function f_X and $g(x)$ is a monotonica increasing and differentiable function from $\mathbb{R}$ to $\mathbb{R}$, show that $Y = g(X)$ the density function

$$f_Y(y) = f_X[g^{-1}(y)]\,\frac{d}{dy}[g^{-1}(y)], \quad y \in \mathbb{R}.$$

(b) Let X be a random variable with exponential distribution with parameter Find the density function of

i. $2X + 5$,

ii. $(1 + X)^{-1}$.

(c) Random variable X and Y have joint density function

$$f_{XY}(x, y) = \begin{cases} k(x^3 + 1)y & \text{if } 0 < x < 1, \quad 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}.$$

Find

i. the value of k,

ii. marginal density functions of X and Y,

iii. $E(XY)$,

iv. Are X and Y independent?

3. (a) Define the Moment Generating Function of a random variable X.

Find the moment generating function of the Gamma dis... given by

$$f(x) = \begin{cases} \dfrac{\lambda^n \, x^{n-1} \, e^{-\lambda x}}{\Gamma(n)} & ; \quad x \geqslant 0 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

Hence find the mean and variance.

(b) i. Define the following terms:

* Unbiased estimator,

* Risk function.

ii. Let $X_1, X_2, ..., X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Determine c such that $c[(X_1 - X_2)^2 + (X_3 - X_4)^2 + (X_5 - X_6)^2]$ is an unbiased estimator for σ^2.

iii. Let $X_1, X_2, ..., X_n$ be a random sample from Poisson distribution with parameter λ. Let $T_1 = \dfrac{X_i + X_j}{2}$ and $T_2 = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. Show that T_1 and T_2 are unbiased estimator for λ and find the best estimator for λ.

Q4. (a) Define the maximum likelihood estimator.

Determine the maximum likelihood estimators of the parameters of the following distributions:

i. Exponential distribution with parameter θ,

ii. Normal distribution with mean μ and variance σ^2.

(b) Let $X_1, X_2, ..., X_n$ be n a random sample from a normal distribution with unknown mean μ and known variance σ^2. Find $100(1-\alpha)\%$ confidence interval for μ.

(c) On the basis of results obtained from a random sample of 100 men from a particular district, the 95% confidence interval for the mean height of the men in the district is found to be (177.22 cm, 179.18 cm). Find the value of $\overline{X}$, the mean of the sample, and σ^2, the standard deviation of the normal population from which the sample is drawn. Calculate the 98% confidence interval for the mean height.