SPECIAL DEGREE EXAMINATION IN MATHEMATICS

2004/2005 (March/April, 2007)

MT 402- MEASURE THEORY

Part II

Answer all questions

Time:Three Hours

This paper consists of 6 questions in a total of 3 pages

1. (a) Let $A \subseteq \mathbb{R}$, with $m^\star(A) < \infty$. Prove that the following four statements are equivalent:

   i. A is measurable;

   ii. $\forall \epsilon > 0, \exists$ open $U \supseteq A$ with $m^\star(U \setminus A) < \epsilon$;

   iii. $\exists\, G \in G_\delta$ with $A \subseteq G$ and $m^\star(G \setminus A) = 0$;

   iv. $\forall \epsilon > 0\ \exists B$, a finite union of open (finite)intervals so that $m^\star(A \triangle B) < \epsilon$.

   (b) Let A be a measurable subset of $\mathbb{R}$, with $m(A) > 0$. Prove that

   $m(A + x) = m(A), \forall x \in \mathbb{R}$;

   ii. there exists a non-measurable subset P of $[0, 1)$;

   iii. If $A^\star = \{x - y : x, y \in A\}$, then $A^\star$ contains an interval $[-\alpha, \alpha]$ for some $\alpha > 0$.

2. Prove that

   (a) if $\{A_n\}_{n=1}^{\infty}$ is an increasing infinite sequence of measurable sets in $\mathbb{R}$, then $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$;

   (b) if $\{A_n\}_{n=1}^{\infty}$ is a decreasing infinite sequence of measurable sets in $\mathbb{R}$ such that $m(A_k) < \infty$ for some k, then $m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n)$;

   (c) if the condition $m(A) < \infty$ is dropped part (b) fails to hold;

   (d) Let A be a measurable subset of $\mathbb{R}$ with $m(A) < \infty$, then the function $x \mapsto m\left(A \cap (-\infty, x]\right)$ is continuous.

3. Let $(X, \mathcal{B}, \mu)$ be a measure space.

(a) What does it to mean to say that a function $f : X \to (-\infty, \infty)$ is $\mathcal{B}$ measurable?

(b) Prove that if $\mathbb{F}$ is a countable, non-void set of such functions f and if
$g(x) = \sup\{f(x) : f \in \mathbb{F}\}$
for each $x \in X$, then g is $\mathcal{B}$ measurable.

(c) Give an example of $X, \mathcal{B}$, and $\mathbb{F}$ to show that the assertion in part (b) can fail if "countable" is omitted.

(d) Let g be an integrable function over a measurable set $A \subseteq \mathbb{R}$. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n(x)| \leq g(x)\ \forall\, x \in A$ and
$\lim_{n\to\infty} f_n(x) = f(x)$ a.e on A. Prove that $\int_A f = \lim_{n\to\infty} \int_A f_n$.
Deduce that $\lim_{n\to\infty} \int_a^\infty \frac{n^2 x e^{-n^2x^2}\,dx}{1+x^2} = 0$, if $a > 0$.
but, the result does not hold if $a = 0$

4. (a) Let (X, Σ, μ) be a measure space, and the completion (X', Σ', μ') of (X, Σ, μ) be defined by $\Sigma' = \{A \cup B | A \in \Sigma, B \subseteq C \text{ for some } C \in \Sigma, \mu(C) = 0\}$ and $\mu'(A') = \mu(A)$ when $A' = A \cup B$.
Prove that (X', Σ', μ') is complete measure space.

(b) Let $(X, \mathcal{B}, \mu)$ be a complete measure space. Let $1 < p < \infty$ and $\mathcal{L}^p(X, \mathcal{B}, \mu)$ comprises all $\mathcal{B}$ measurable functions f on X for which
$\int_X |f|^p d\mu < \infty$, and $||f||_p = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}$ for $f \in \mathcal{L}^p(X, \mathcal{B}, \mu)$.
Prove that

i. if $f, g \in \mathcal{L}^p(X, \mathcal{B}, \mu)$, then $f + g \in \mathcal{L}^p(X, \mathcal{B}, \mu)$ and $||f+g||_p \leq ||f||_p + ||g||_p$;

ii. $\mathcal{L}^p(X, \mathcal{B}, \mu)$ with $||.||_p$ is a complete normed linear space.

5. Let $(X, \mathcal{B}, \mu)$ be a measure space. Prove that

(a) of $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of non-negative measurable functions on X, with $\lim_{n\to\infty} f_n = f$, then $\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$, but, the result does not hold for decreasing sequences.

(b) if $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-negative measurable functions on X, with $\lim_{n\to\infty} f_n = f$ a. e, then $\int_X f d\mu \leq \lim_{n\to\infty} inf \int_X f_n d\mu$, however strict inequality may not hold.

(c) if f, g are two non-negative measurable functions on X, and let a, b be non-negative constants, then $af + bg$ is measurable and
$\int_X (af + bg) d\mu = a \int_X f d\mu + b \int_X g d\mu$

(d) if $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-negative measurable functions on X, then
$\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$

(e) If

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } 0 < x < \infty, \\ 1, & \text{if } x = 0 \end{cases}$$

then $L \int_0^{\infty} f d\mu$ does not exist.

6. (a) State and prove Fubini's theorem for $f \in \mathcal{L}(X \times Y, \Sigma_X \times \Sigma_Y, \mu \times \nu)$(Tonelli's theorem may be assumed)

(b) Prove that if $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is defined by

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{(x^2 + y^2)^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

then the iterated integrals are not equal. Is f integrable?

(c) By considering $\int_0^a \int_0^{\infty} e^{-xt} \sin x dt dx$

prove that $\lim_{a\to\infty} \int_0^a \frac{\sin x}{x} dx = \frac{\pi}{2}$