EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION

IN MATHEMATICS, (2004/2005)

(MARCH/APRIL, 2007)

PART II

MT 405 - NUMERICAL THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Answer all Questions Time allowed: Three hours

1. (a) Let (x_n) be a sequence of real numbers satisfying

$$x_{n+1} \leq \frac{1}{1-A}(x_n + AB), \quad n = 0, 1, 2, \cdots,$$

where $0 \leq A < 1$ and $B \geq 0$. Prove that

$$x_n \leq \frac{1}{(1-A)^n}\, x_0 + \left[\frac{1}{(1-A)^n} - 1\right] B, \quad n = 0, 1, 2, \cdots,$$

and deduce that

$$x_n \leq e^{na} x_0 + (e^{na} - 1)B, \quad a = \frac{A}{1-A}, \quad n = 0, 1, 2, \cdots$$

(b) Let y be the continuous solution of an m- dimensional system

$$y'(x) = f(y(x)), \quad y(0) = \nu,$$

where for some norm

$$||f(u) - f(v)|| \leq L||u - v|| \quad \text{and} \quad ||f(u)|| \leq M$$

for all $u, v \in \mathbb{R}^m$. Use the identity

$$y(x+h) - y(x) - hy'(x+h) = h\int_0^1 [y'(x+ht) - y'(x+h)]dt$$

to show that , for any x and h ,

$$||y(x+h) - y(x) - hy'(x+h)|| \leq \frac{h^2}{2} LM .$$

(c) For given y_0 , let $y_1, y_2, \cdots, y_N$ be given by the implicit Euler method

$$y_{n+1} = y_n + hf(y_{n+1}), \qquad n = 0, 1, \cdots, N-1,$$

where h is chosen so that $hN = 1$.

Show that , for $hL < 1$,

$$||y(1) - y_N|| \le e^{\frac{L}{1-hL}} ||y(0) - y_0|| + \frac{h}{2}\left(e^{\frac{L}{1-hL}} - 1\right) M$$

and comment briefly on this result.

2. (a) Define the following terms:

i. Convergence,

ii. Consistency,

iii. Zero Stability,

applied to the linear multi-step method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{h} \beta_j f_{n+j}, \qquad \alpha_k = 1,$$

used for solving initial value problem of the form

$$y' = f(x, y), \qquad a \le x \le b, \quad y(a) = \nu,$$

where $y : [a, b] \longrightarrow \mathbb{R}^m$ and $f : [a, b] \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$.

What is the relation between these terms ?

Prove that if a linear multi-step method is convergent , then it is zero-stable .

(b) Find the range of values of α for which the linear 3-step method

$$y_{n+3} + \alpha(y_{n+2} - y_n) - y_{n+1} = \frac{1}{2}(3 + \alpha)h(f_{n+2} + f_{n+1})$$

is zero stable. Show that this method is not convergent for these values of α .

3. (a) i. Define the order of the linear multi-step method in terms of the associated linear operator.

ii. Determine the linear 2-step method of maximum order.

(b) i. Show that a linear multi-step method with characteristic polynomials ρ and σ is of order p if and only if

$$\rho(z) - (\ln z)\sigma(z) = c_{p+1}(z-1)^{p+1} + c_{p+2}(z-1)^{p+2} + \cdots,$$

$$|z-1| < 1, \text{ with } c_{p+1} \neq 0.$$

ii. A linear multi-step method with characteristic polynomial

$$p(z) = z^2 - \frac{3}{2}z + \frac{1}{2}$$

is of maximum order. Find the method and the error constant. Explain why the method is convergent.

4. (a) Define the term "absolute stability " as applied to a numerical method used for solving initial value problems for ordinary differential equations.

(b) A linear multi-step method has characteristic polynomials ρ and σ. Show that the method is absolutely stable for given $z \in \mathbb{C}$ if and only if the zeros of $\rho(r) - z\sigma(r)$ are of modulus at most one, with zeros of modulus one being simple.

(c) The explicit Euler method is used as predictor and the Trapezoidal rule is used as corrector in the PEC mode. Show that the combined method is absolutely stable for given $z \in \mathbb{C}$ if the roots of $r^2 - (1+\frac{3z}{2})r + \frac{1}{2}z$ are of modulus at most one with roots of modulus one being simple. Show that the method is absolutely stable for real $z \in [-1, 0]$.

5. (a) i. The coefficient of an s-stage Runge-Kutta method are given by the array

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array}, \quad C = Ae\,,\ e = (1, 1, \cdots, 1)^T.$$

Show that the method is absolutely stable for given z if $det(I - zA) \neq 0$ and $|R(z)| \leq 1$, where $R(z) = 1 + zb^T(I - zA)^{-1}e$.

ii. Deduce that , for an explicit method , $R(z)$ is a polynomial of degree s and hence prove that all explicit s-stage Runge-Kutta methods of order s have identical regions of absolute stability .

(b) Show that the 3-stage Runge-Kutta method with coefficients

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 2/3 & 2/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline & 1/4 & 1/6 & 7/12 \end{array}$$

is of order 2 and that the interval of absolute stability is $[-3, 0]$.

6. (a) i. Define the term "B-stability" as applied to an s-stage Runge-Kutta method given by the array

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array}, \quad C = Ae\,,\ e = (1, 1, \cdots, 1)^T\,, b^T = (b_1, b_2, \cdots, b_j)\,.$$

ii. Let $B = diag(b_1, b_2, \cdots, b_j)$ and

$$Q = BA^{-1} + A^{-T}B - A^{-T}bb^TA^{-1}.$$

Prove that if B and Q are non-negative definite, then the Runge-Kutta method is B-stable .

(b) i. Define what is meant by the statement that a Runge-Kutta method is algebraically stable . State the relationship between B-stability and algebraic stability .

ii. Prove that the one parameter family of semi-implicit methods given by the array

$$
\begin{array}{c|cc}
\frac{3+\alpha}{6} & \frac{3+\alpha}{6} & 0 \\
\frac{\alpha-1}{2\alpha} & -\frac{1}{\alpha} & \frac{\alpha+1}{2\alpha} \\
\hline
 & \frac{3}{3+\alpha^2} & \frac{\alpha^2}{3+\alpha^2}
\end{array}
$$

is algebraically stable for all $\alpha > 0$.