EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2007/2008

FIRST SEMESTER(Dec./Jan., 2008/2009)

MT 306 - PROBABILITY THEORY

PROPER AND REPEAT

Answer all questions | Time: Two hours

1. (a) Define the term "conditional probability".

Let A, B and C be three events such that $P(C) > 0$. Prove that

i. $P(A^c|C) = 1 - P(A|C)$,

ii. $P[(A \cup B)|C] = P(A|C) + P(B|C) - P[(A \cap B)|C]$.

(b) Let X have a geometric distribution with parameter p.

i. Prove that $P[X \leq n] = 1 - q^n$, where n is a positive integer and $q = p - 1$. Hence deduce that $P[X > n] = q^n$.

ii. Show that for any positive integers m and n,

$$P[X > m + n \mid X > m] = P[X > n].$$

(c) A coin is biased, so that the probability of obtaining a head is 0.6. If X is the number of tosses up to and including the first head, find the following:

i. $P[X \leq 4]$,

ii. $P[X > 5]$,

iii. the probability that more 8 tosses will be required to obtain a head given that more than 5 tosses are required.

2. Define the term "Moment generating function" of a random variable X.

   (a) Show that if X and Y are independent random variables, then $X+Y$ has the moment generating function,

   $$M_{X+Y}(t) = M_X(t)M_Y(t),$$

   where M_X and M_Y are moment generating functions of X and Y, respectively and t is a real variable.

   (b) The probability density function of a Gamma distribution with parameters m and λ is given by

   $$f_X(x) = \begin{cases} \dfrac{\lambda^m x^{m-1} e^{-\lambda x}}{\Gamma(m)} & \text{if } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

   Let X and Y be independent random variables; X having the gamma distribution with parameters m and λ and Y having the gamma distribution with parameters s and λ. Show that $X+Y$ has the gamma distribution with parameters $m+s$ and λ.

   (c) Show that the ψ^2 distribution with n degrees of freedom has moment generating function

   $$M(t) = (1-2t)^{-n/2}\,, \quad \text{if } t < \frac{1}{2}$$

   by using the result of (b).

   (d) The random variable X follows the normal distribution with mean 0 and variance 1. Find the moment generating function of X^2.

   Deduce that, if $X_1,\ X_2, \cdots, X_n$ are independent random variables having the normal distribution with mean 0 and variance 1, then $U = X_1^2 + X_2^2 + \cdots + X_n^2$ has a ψ^2 distribution with n degrees of freedom.

3. (a) If X is a continuous random variable with density function f_X and g is monotonically increasing and differentiable function from $\mathbb{R}$ into $\mathbb{R}$, show that $Y = g(X)$ has the density function

$$f_Y(y) = f_X\left[g^{-1}(y)\right] \frac{d}{dy}\left[g^{-1}(y)\right], \qquad y \in \mathbb{R}.$$

If the probability density function of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{kx^3}{(1+2x)^6} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Find the probability density function of the random variable $Y = \dfrac{2X}{1+2X}$.

(b) Let the random variables X and Y have the joint probability density function

$$f_{XY}(x,y) = \begin{cases} e^{-x-y} & \text{if } x,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

and let $U = X + Y$ and $V = \dfrac{X}{X+Y}$.

i. Find the joint probability density function of U and V.

ii. Are U and V independent random variables?

4. (a) Let $X_1,\ X_2, \cdots, X_n$ be the random samples from the normal distribution with mean 0 and variance σ^2. Find the maximum likelihood estimators of μ and σ^2.

(b) Let $X_1,\ X_2, \cdots, X_n$ be the random samples from the normal distribution with mean 0 and variance θ, $0 < \theta < \infty$. Show that

$$T = \frac{\sum_{i=1}^{n} X_i^2}{n}$$

is an unbiased estimator for θ. Also show that, by the Crammer-Rao inequality, $V(T) \geq \dfrac{2\theta^2}{n}$, where $V(T)$ is variance of T.

(c) The sample mean and sample variance of 6 observations from the first population are 35 and 42 respectively and those of 10 observations from the second population are 40 and 28 respectively. Construct the 90% confidence interval for the difference of sample mean $\mu_1 - \mu_2$.