

EASTERN UNIVERSITY, SRI LANKA

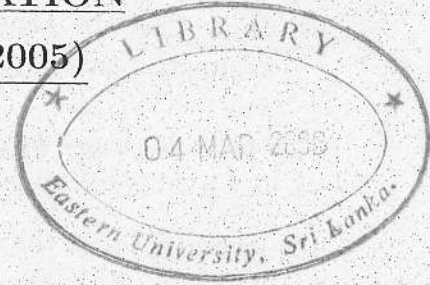
SPECIAL DEGREE EXAMINATION

IN MATHEMATICS, (2004/2005)

(MARCH/APRIL, 2007)

PART I

MT 408 - RELATIVITY



Answer all questions

Time allowed: 3 Hours

- Q1. (a) A number of point charges e_1, e_2, \dots, e_N are fixed at interior points Q_1, Q_2, \dots, Q_N of a line OX . Show that if P is a point on any selected line of force, then

$$\sum_{i=1}^N e_i \cos \theta_i = \text{constant, where } \theta_i = P\hat{Q}_iX.$$

- (b) Positive charges q_1 and q_2 are placed at points A and B respectively. Consider the line of force starting from A at an angle α to BA . Prove that its asymptote passes through the point C on AB such that $\frac{AC}{CB} = \frac{q_2}{q_1}$ and makes an angle β with BA given by

$$\sin\left(\frac{\beta}{2}\right) = \left(\frac{q_1}{q_1 + q_2}\right)^{\frac{1}{2}} \sin\left(\frac{\alpha}{2}\right).$$

- (c) Show that the electric field potential due to dipole \mathbf{P} is

$$\frac{1}{4\pi\epsilon_0} \frac{\mathbf{P} \cdot \mathbf{r}}{r^3},$$

where \mathbf{r} is the vector from a dipole to the point concerned.

- Q2. In spherical coordinates, the Laplace equation is given by:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v}{\partial \phi^2} = 0.$$

- (a) Show that the general solution of the Laplace equation in the axially symmetric case is given by:

$$v(r, \theta) = \sum_{n=0}^{\infty} (C_n r^n + D_n r^{-(n+1)}) P_n(\cos \theta),$$

where $P_n(x)$ is the Legendre polynomial of degree n .

- (b) An earthed conducting sphere of radius a is coated with a thickness $b - a$ of dielectric of dielectric constant k . The sphere and dielectric are placed in a uniform electric field E :

- (i) Show that the change in the field outside the dielectric is the same as that produced by an electric dipole of moment

$$4\pi\epsilon_0 E b^3 \left(\frac{(2k+1)a^3 + (k-1)b^3}{2(k-1)a^3 + (k+2)b^3} \right)$$

at the centre of the sphere.

- (ii) Show also that the surface density of charge at a point on the conductor is

$$\frac{qk\epsilon_0 E \cos \theta}{k+2+2(k-1)\frac{a^3}{b^3}}$$

where θ is the angle between the radius to the point and the direction of the field.

- Q3. (a) Discuss the significance of

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dv'$$

in connection with Ampere's Law, giving the force between two current carrying loops of steady current.

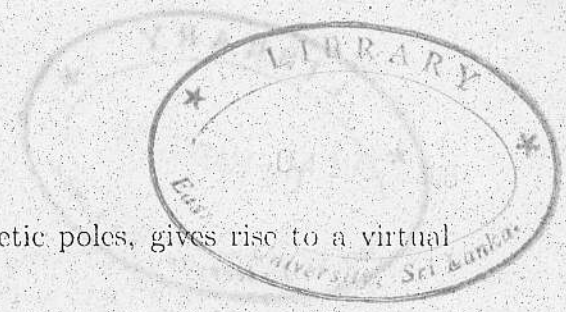
- (b) Prove that:

(i) $\text{curl } \mathbf{B} = \mu_0 \mathbf{j}$;

(ii) $\mathbf{B} = \text{curl } \mathbf{A}$, where $\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv'$;

(iii) $\text{div } \mathbf{A} = 0$;

(iv) $\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \mathbf{m} \wedge \nabla \left(\frac{1}{r} \right)$ for a magnetic pole of strength \mathbf{m} at the origin.



(v) volume distribution $\mathbf{m}(\mathbf{r})$ of magnetic poles, gives rise to a virtual current distribution $\nabla \wedge \mathbf{m}(\mathbf{r})$.

(c) Discuss the generalisation of $\text{curl } \mathbf{B} = \mu_0 \mathbf{j}$ to unsteady currents.

Q4. In two spacetime dimensions two observers moving with constant relative velocity v set up inertial frames \mathcal{R} and \mathcal{R}' with coordinate systems (ct, x) and (ct', x') respectively.

(a) Starting with a linear transform and the invariance of the speed of light, show that if they set their clocks to $t = t' = 0$ when they pass each other, the transform between these coordinate systems is the Lorentz transform:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}, \quad \text{where } \gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

(b) For the following you should state any results you use but do not have to prove them:

(i) Show that a particle moving with the speed of light in \mathcal{R} also moves at the speed of light in \mathcal{R}' .

(ii) Two events are simultaneous in \mathcal{R} , but occur one second apart in \mathcal{R}' . Calculate the velocity in terms of c of \mathcal{R}' relative to \mathcal{R} if the distance between the event is 10^6 km in \mathcal{R} .

(iii) A vehicle travels with speed $0.1c$ in \mathcal{R}' . How fast is it travelling in \mathcal{R} ??

(iv) An object which is stationary in \mathcal{R} has length 2m . How long does it appear in \mathcal{R}' ?

Q5. Two electrons with rest mass m_0 each have energy E in the centre-of-mass frame.

(a) Show that in the laboratory frame in which one electron is originally at rest, the other has initial energy $(2E^2 - m_0^2 c^4) / (m_0 c^2)$.

(b) The electrons collide elastically and then move at right-angles to their original directions, as measured in the centre-of-mass frame. Find the angle between the electrons after the collision as measured in the laboratory frame, and their new energy.

- (c) The experiment is repeated, but this time after the collision there are two electrons and a π^0 meson (with rest mass m_0^π). What is the minimum velocity of the moving electron required for this to occur?

- Q6. (a) (i) Define the term 4-vector. What is the 4-momentum P of a massless particle (such as a photon)?
 (ii) Define the inner product $g(X, Y)$ for two 4-vectors and hence show that P is null. Does this hold for massive particles (such as electrons)?
- (b) Consider light emitted at an angle θ' in the rest frame \mathcal{R}' of a source moving with speed v .

- (i) Show that the light has an observed angle θ satisfying:

$$\tan \theta = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \frac{\sin \theta'}{\cos \theta' + \frac{v}{c}}$$

- (ii) Furthermore show that a photon with energy E' in the rest frame \mathcal{R}' of the source has observed energy E , where

$$E = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} E' \left(1 + \frac{v}{c} \cos \theta'\right)$$

- (iii) Show that if $\frac{v}{c}$ is close to unity then any forward shining light ($-\frac{\pi}{2} \leq \theta' \leq \frac{\pi}{2}$) is observed to be concentrated in a narrow cone whose semi-angle θ is given by $\sin \theta = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$.