EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2005/2006)

SECOND SEMESTER (Dec. '2008/Jan. '2009)

MT 301 - GROUP THEORY

Repeat (Special)

Answer all questions　　　　　　　　　　　　　　Time: Three hours

1. (a) Define the following terms

   i. group;

   ii. cyclic group;

   iii. abelian group.

   Prove that every subgroup of a cyclic group is cyclic.

   Is the converse part true? Justify your answer.

   (b) State and prove Lagrange's theorem.

   i. In a group G, H and K are different subgroups of order p, p is prime. Show that $H \cap K = \{e\}$, where e is the identity element of G.

   ii. Prove that in a finite group G, the order of each element divides order of G. Hence prove that $x^{|G|} = e$, $\forall\, x \in G$.

2. (a) What is meant by saying that a subgroup of a group is normal?

   i. Let H and K be two normal subgroups of a group G. Prove that $H \cap K$ is a normal subgroup of G.

   ii. Prove that every subgroup of an abelian group G is a normal subgroup of G.

(b) With usual notations prove that

i. $N(H) \leq G$;

ii. $H \trianglelefteq N(H)$.

(c) Let $Z(G) = \{x \in G \mid xg = gx, \forall\ g \in G\}$. Prove the following

i. $Z(G) = \bigcap_{a \subset G} C(a)$, where $C(a) = \{g \in G\ :\ ga = ag\}$

ii. $Z(G) \trianglelefteq G$.

3. (a) State the first isomorphism theorem.

Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove the following

i. $K \trianglelefteq H$;

ii. $H/K \trianglelefteq G/K$;

iii. $\dfrac{H/K}{G/K} \cong G/H$.

(b) Write down the class equation of a finite group G.

Let G be a group of order p^n, where p is a prime number. Prove that, $Z(G) = G$ if $n = 2$.

4. (a) Define commutator subgroup G' of a group G.

Prove that the following

i. $G' \trianglelefteq G$;

ii. G/G' is abelian.

(b) Let $H \trianglelefteq G, P = \{K \leq G\ :\ H \subseteq K\}$ and $Q = \{K'\ :\ K' \leq G/H\}$.

Prove that there exists a one to one correspondence between P and Q.

5. (a) What is meant by the "internal direct product" as applied to a group.

Is it true that all the groups satisfy the internal direct product property? Justify your answer.

Let H and K be two subgroups of a group G, prove that G is a direct product of H and K if and only if

i. each $x \in G$ can be uniquely expressed in the form $x = hk$, where $h \in H, k \in K$.

ii. $hk = kh$ for any $h \in H, k \in K$.

(b) Define the term "$p-$group".

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

6. (a) Define the following terms as applied to a group.

i. Permutation;

ii. Cycle of order r;

iii. Transposition.

(b) Prove that the permutation group on n symbols (S_n) is a finite group of order $n!$.

Is it true that S_n is abelian for $n > 2$? Justify your answer.

(c) Prove that every permutation in S_n can be expressed as a product of transpositions.

(d) Prove that the set of even permutations forms a normal subgroup of S_n.