EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2009/2010

FIRST SEMESTER (June/July' 2011)

MT 201 - VECTORSPACES AND MATRICES

Answer all question — Time: Three hours

1. (a) Define what is meant by

   i. a vector space;

   ii. a subspace of a vector space.

   (b) Let $V = \{x : x > 0, x \in \mathbb{R}\}$. Define addition "$\oplus$" and scalar multiplication "$\odot$" on V as follows:

$$x \oplus y = xy,$$

$$r \odot x = x^r,$$

   $\forall\, r \in \mathbb{R}$ and $\forall\, x, y \in V$. Prove that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

   Let

$$x \oplus y = xy,$$

$$r \odot x = rx,$$

   $\forall\, r \in \mathbb{R}$ and $\forall\, x, y \in V$. Is $(V, \oplus, \odot)$ a vector space over $\mathbb{R}$? Justify your answer.

   (c) Let M be a vector space of 2×2 matrices over $\mathbb{R}$. Which of the following subsets are subspaces of M? Justify your answer.

   i. set of all 2×2 matrices with zero determinant;

   ii. set of all 2×2 idempotent matrices.

2. (a) State the dimension theorem for two subspaces of a finite dimensional vector space.

(b) Let V be a finite dimensional vector space with the usual notations. Prove the following:

i. if $dimV = n$, then there exist one dimensional subspaces $U_1, U_2, \cdots, U_n$ of V such that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$.

ii. if $U_1, U_2, \cdots, U_m$ are subspaces of V, then

$$dim(U_1 + U_2 + \cdots + U_m) \leq dimU_1 + dimU_2 + \cdots + dimU_m.$$

(c) i. Prove that if $\{v_1, v_2, \cdots, v_n\}$ spans V, then so does the tuple $\{v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n\}$.

ii. Let V be a vector space of $\mathbb{R}^5$ defined by

$$V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of V.

iii. If U_1 and U_2 are both 5 dimensional subspaces of $\mathbb{R}^9$, then prove that $U_1 \cap U_2 \neq \{0\}$.

3. (a) Define the following:

i. range space $R(T)$;

ii. null space $N(T)$

of a linear transformation T from a vector space V into another vector space W.

(b) Find $R(T)$ and $N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z), \forall\ (x, y, z) \in \mathbb{R}^3.$$

Verify the equation $dim\ V = dim(R(T)) + dim(N(T))$ for this linear transformation.

(c) i. Let $\mathbb{P}_3$ be the set of all polynomials of degree ≤ 3 and let $T : \mathbb{R}^3 \to \mathbb{P}_3$, be a linear transformation defined by

$$T(x_1, x_2, x_3) = x_1 + (x_2 + x_3)x + (x_3 - x_1)x^2 + x_3\ x^3.$$

Find the matrix representation of T with respect to the bases $B_1 = \{(1,1,1),(1,2,3),(2,-1,1)\}$ and $B_2 = \{1+x, x+x^2, x^2+x^3, x^3\}$ of $\mathbb{R}^3$ and $\mathbb{P}_3$, respectively.

ii. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x,y,z) = (x + 2y - z, y + z, x + y - 2z)$ and let $B_1 = \{(0,0,1),(0,1,1),(1,1,1)\}$ and $B_2 = \{(1,1,0),(0,1,1),(1,0,1)\}$ be bases for $\mathbb{R}^3$. Find the matrix representation of T with respect to the basis B_2 by using the transition matrix.

4. (a) Define the following terms as applied to a matrix:

   i. rank;

   ii. echelon form;

   iii. row reduced echelon form.

   (b) Let A be an $n \times n$ matrix. Prove the following:

   i. row rank of A is equal to column rank of A;

   ii. if B is an $n \times n$ matrix obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

   (c) i. Find the row rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

   ii. Find the row reduced echelon form of the matrix

$$\begin{pmatrix} -1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}.$$

5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

   i. cofactor A_{ij} of an element a_{ij};

   ii. adjoint of A ($adj\ A$).

With the usual notations, prove that

$$A.(adj\ A) = (adj\ A).A = det\ A.\ I.$$

Hence prove $adj(adj\ A) = (det\ A)^{n-2}A$.

(State any results you may use)

(b) Let P be an $n \times n$ matrix with all elements are equal to α ($\in \mathbb{R}$). For any non-zero scalar $\mu \in \mathbb{R}$, prove that

i. $det(P + \mu I) = \mu^{n-1}(n\alpha + \mu)$;

ii. $$(P+\mu I)^{-1} = \frac{1}{\mu(n\alpha+\mu)} \begin{pmatrix} (n-1)\alpha+\mu & -\alpha & \cdots & -\alpha \\ -\alpha & (n-1)\alpha+\mu & \cdots & -\alpha \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -\alpha & -\alpha & \cdots & (n-1)\alpha+\mu \end{pmatrix}$$

6. State the necessary and sufficient condition for a system of linear equations to be consistent.

7. (a) Suppose n is a positive integer and $a_{i,j} \in \mathbb{R}$ for $i, j = 1, 2, \cdots, n$. Prove that the following are equivalent:

i. the trivial solution $x_1 = x_2 = \cdots = x_n = 0$ is the only solution to the homogeneous system

$$\sum_{k=1}^{n} a_{1,k}x_k = 0,$$

$$\sum_{k=1}^{n} a_{2,k}x_k = 0,$$

$$\vdots$$

$$\sum_{k=1}^{n} a_{n,k}x_k = 0.$$

ii. for every constant, $c_1, c_2, \cdots, c_n \in \mathbb{R}$, there exists a solution to the system of equations

$$\sum_{k=1}^{n} a_{1,k}x_k = c_1,$$

$$\sum_{k=1}^{n} a_{2,k}x_k = c_2,$$

$$\vdots$$

$$\sum_{k=1}^{n} a_{n,k}x_k = c_n.$$

(b) Investigate for what value of λ, μ the system of liner equation

$$x + y + z = 6,$$

$$x + 2y + 3z = 10,$$

$$x + 2y + \lambda z = \mu,$$

have

i. no solution;

ii. a unique solution;

iii. an infinite number of solutions.

(c) A bag contains 3 types of coins, namely, Rs.1, Rs.2 and Rs.5. There are 30 coins amounting to Rs.100 in total. Find the number of coins in each category.