



EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - (2009/2010)

FIRST SEMESTER (June/July, 2011)

MT 203 - EIGENSPACES AND QUADRATIC FORMS

Answer all questions

Time: Two hours

1. (a) Define the following terms as applied to a square matrix $A = (a_{ij})$:
- eigenvalue;
 - characteristic polynomial, $\psi_A(\lambda)$, of A ;
 - trace of A ($tr(A)$).
- (b) Let x be an eigenvector of a real $n \times n$ matrix A corresponding to the eigenvalue λ . Show that x is an eigenvector corresponding to the eigenvalue λ^m of A^m , for each $m = 1, 2, 3, \dots$. Hence show that, if A is
- an idempotent matrix, then λ must be 0 or 1.
 - a nilpotent matrix, then $\psi_A(t) = t^n$ and $tr(A) = 0$.
- (c) Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of an $n \times n$ matrix A with multiplicities. Prove the following:
- $\lambda_j = a_{jj} + \sum_{i \neq j} (a_{ii} - \lambda_i)$, for $j = 1, 2, \dots, n$;
 - $\det A = \lambda_1 \times \lambda_2 \times \dots \times \lambda_n$, where $\det A$ means determinant of A .
- (d) Prove that, if two diagonalizable matrices A and B have the same eigenvectors then, $AB = BA$.
- Prove the converse of the above statement with an assumption that the eigenvalues of A are all distinct.

2. (a) Define the following terms:

- i. minimum polynomial;
- ii. irreducible polynomial,

of a square matrix.

(b) Prove the following:

- i. If $m(t)$ is the minimum polynomial of an $n \times n$ matrix A and $\psi_A(t)$ is characteristic polynomial of A , then $\psi_A(t)$ divides $[m(t)]^n$.
- ii. The characteristic and minimum polynomials of a square matrix have same irreducible factors.
- iii. $f(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0$ is the minimum polynomial of n -square matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

Hence find the matrix whose minimum polynomial is $t^4 - 5t^3 - 2t^2 + 7t$.

3. (a) Find an orthogonal transformation which reduces the following quadratic form to a diagonal form

$$5x_1^2 + 11x_2^2 - 2x_3^2 + 12x_1x_3 + 12x_2x_3.$$

(b) Simultaneously diagonalize the following pair of quadratic forms

$$x_1^2 - x_2^2 + x_3^2 - 2x_2x_3 - 2x_1x_3 - 2x_1x_2;$$

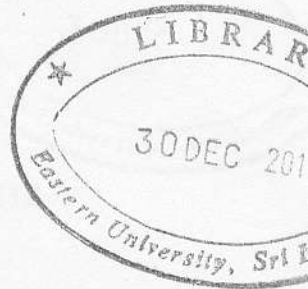
$$3x_1^2 + x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_2x_3 - 2x_1x_3.$$

4. (a) Prove that A is an $n \times n$ real symmetric matrix if and only if there exists an orthogonal matrix Q such that $Q^T A Q$ is diagonal.

Find an orthogonal matrix Q and a diagonal matrix D such that $Q^T A Q = D$,

where

$$A = \begin{pmatrix} -2 & 4 & -2 \\ 4 & 4 & -4 \\ -2 & -4 & 5 \end{pmatrix}.$$



- (b) Define the term inner product in a vector space.

Let $C[0, 1]$ be the vector space of all real-valued continuous functions on $[0, 1]$.

For any two functions $f(x)$ and $g(x)$ in $C[0, 1]$, define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Show that $\langle \cdot, \cdot \rangle$ is an inner product on $C[0, 1]$.

- (c) Use the Gram-Schmidt process to find orthonormal basis for the column space of the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix}.$$