EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2009/2010

FIRST SEMESTER - (June/July, 2011)

MT 302 - COMPLEX ANALYSIS

PROPER & REPEAT

Answer **all** questions   Time allowed: **3 Hours**

Q1. (a) Let $(z_n)_{n=1}^{\infty}$ be a sequence of points in $\mathbb{C}$. Define what is meant by $z_n \to z$ as $n \to \infty$. **[5 marks]**

Let $z \in \mathbb{C}$. Prove that if $|z| < 1$, then

$$1 + z + z^2 + \ldots = \frac{1}{1-z}.$$

(You may use the result that if $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$.)

Hence show that if $|a| < 1$, then

$$1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \ldots = \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}.$$

[30 marks]

(b) Let $f : A \subseteq \mathbb{C} \to \mathbb{C}$ be differentiable at some $z_0 = x_0 + i\,y_0 \in A$. If $f(z) = u(x,y) + i\,v(x,y)$, then prove that $u(x,y)$ and $v(x,y)$ have partial derivatives at $z_0 = x_0 + i\,y_0$ that satisfy the *Cauchy-Riemann* equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

[40 marks]

(c) (i) Define what is meant by the function $h : \mathbb{R}^2 \to \mathbb{R}$ being *harmonic*. **[5 marks]**

(ii) Let $u(x,y)$ and $v(x,y)$ be harmonic functions in a domain D. Show that the function given by

$$f(z) := \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

is analytic in D. **[20 marks]**

Q2. (a) (i) Define what is meant by a *path* $\gamma : [\alpha, \beta] \to \mathbb{C}$. **[5 marks]**

(ii) For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)\, dz$. **[5 marks]**

(b) (i) Let $D(a;r) := \{z \in \mathbb{C} : |z-a| < r\}$ denote the open disc with center $a \in \mathbb{C}$ and radius $r > 0$ and let f be analytic on $D(a;r)$ and $s \in (0,r)$. Prove *Cauchy's Integral Formula*, that is for $z_0 \in D(a;s)$,

$$f(z_0) = \frac{1}{2\pi i} \int_{C(a;\,s)} \frac{f(z)}{z - z_0}\, dz,$$

where $C(a;s)$ denotes the circle with center a and radius $s > 0$. **[40 marks]**

(ii) Let $C(0;1)$ be the unit circle $z = e^{it}$, $-\pi \leqslant t \leqslant \pi$. Show that for any real constant a,

$$\int_{C(0;1)} \frac{e^{az}}{z}\, dz = 2\pi i\,.$$

[20 marks]

Hence deduce that

$$\int_0^\pi e^{a\cos t} \cos(a \sin t)\, dt = \pi\,.$$

[30 marks]

Q3. (a) Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being *entire*.

[5 marks]

(b) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$ be an analytic function. Prove that

(i) f has a Taylor series expansion about z_0, that is, there exists a unique sequence $(a_n)_{n=0}^{\infty} \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in D(z_0; r) \subseteq A,$$

and the coefficient a_n is given by

$$a_n = \frac{1}{2\pi i} \int_{C(z_0;\, s)} \frac{f(t)}{(t - z_0)^{n+1}}\, dt, \quad n \geqslant 0, \quad 0 < s < r.$$

[25 marks]

(ii) f has derivatives of all orders on A. [25 marks]

(iii) $f^{(n)}(z_0) = n!\, a_n$. [15 marks]

(c) Using the results in (b), expand $f(z) = \ln(1 + z)$ in Taylor series about the point $z = 0$. [30 marks]

Q4. (a) State the *Maximum-Modulus Theorm.* [10 marks]

Suppose that the function $J(z) = u(x, y) + iv(x, y)$ is analytic everywhere in the xy-plane and $u(x, y)$ bounded for all (x, y) in the xy-plane. Using the Maximum-Modulus Theorm, prove that $u(x, y)$ is constant throughout the plane. [20 marks]

(b) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \to \mathbb{C}$, where $D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}$. Define what is meant by

(i) the *order* of f at z_0; [5 marks]

(ii) f having a *pole* at z_0 of order m. [5 marks]

(c) (i) State the *Residue Theorem.* [10 marks]

(ii) If f has a pole of order m at z_0, then prove that

$$Res(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} h^{(m-1)}(z),$$

where $h(z) = (z - z_0)^m f(z)$. [30 marks]

Using (i) and (ii), evaluate

$$\frac{1}{2\pi i} \int_{C(0;3)} \frac{e^{tz}}{z^2(z^2 + 2z + 2)} dz,$$

where t is a real parameter. [20 marks]

Q5. Let $\alpha > 0$ and suppose that

(i) f is analytic in the upper-half plane, that is, $\{z : \text{Im}(z) \geq 0\}$, except possibly for finitely many singularities, none are on the real axis.

(ii) $f(z) \to 0$ as $|z| \to \infty$ with $\text{Im}z \geq 0$.

Then prove that

$$I := \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx$$

exists and

$$I = 2\pi i \times \text{sum of residues of } e^{i\alpha z} f(z) \text{ in the upper} - \text{half plane.}$$

[60 marks]

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2}\, dx.$$

[40 marks]

Q6. (a) State the *Principle of the Argument Theorem.*

[10 marks]

(b) Prove *Rouche's Theorem*: Let γ be a simple closed path in an open starset A. Suppose that

(i) f, g are analytic in A except for finitely many poles, none lying on γ.

(ii) f and $f+g$ have finitely many zeros in A.

(iii) $|g(z)| < |f(z)|, \quad z \in \gamma$. Then

$$ZP(f+g;\gamma) = ZP(f;\gamma)$$

where $ZP(f+g;\gamma)$ and $ZP(f;\gamma)$ denote the number of zeros − number of poles inside γ of $f+g$ and f, respectively, where each is counted as many times as its order.

[50 marks]

(c) Prove that all the zeros of the equation $R(z) = z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. **[40 marks]**